Cobweb Model in System Dynamics Form
Hyperincursive Perspective

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Abstract

The cobweb model of competitive market dynamics has been examined in the form of system dynamics model. Separation of the structure elements and introduction of anticipative hyperincursive algorithm was used for transformation of the classical cobweb model to the accelerator based one. The cyclical response of the system that depends on the demand~supply parameters and eigenvalues of the characteristic equation has been numerically examined. The concept of parameter differentiation and time response of the system is transformed to the periodicity concept where periodicity is the main, driven property of the model. As such this is the key attribute in complex discrete agent-based adaptive anticipatory models. The periodic conditions of the model has been analytically determined by the application of *z-transform*. The periodicity conditions of the initial map have been preserved in the nonlinear case. By the application of the Lyapunov exponents several stability regions of the nonlinear model were numerically determined.

Keywords: cobweb, hyperincursion, system dynamics, anticipative system, nonlinear system, period, Farey tree, chaos

1 Introduction

Cobweb model presents the market demand-supply adjustment. It is typically viewed as the model of the agricultural pricing mechanism. The story behind the model is briefly explained as the common agriculture market adjustment mechanism: "The quantity offered for sale this year depends on what was planted at the start of the growing season, which in turn depends on the last year's price. Consumers look at the current prices, though, when deciding to buy. The cobweb model also assumes that the market is perfectly competitive and that supply and demand are both linear schedules." For clear and extensive introduction to the topic one should see [1, 2]. The fact that the studied cobweb model is in the field of discrete dynamics [3] is rather an advantage since the systems of difference equations are often easier to grasp. For example in his enduring scholarly value work on the studies of Dynamic Systems Luenberger [4] on the first place addresses difference and later on differential equations. The model in question has all the characteristics of classical System Dynamics (SD) models: equilibrium, competitiveness, human perception,
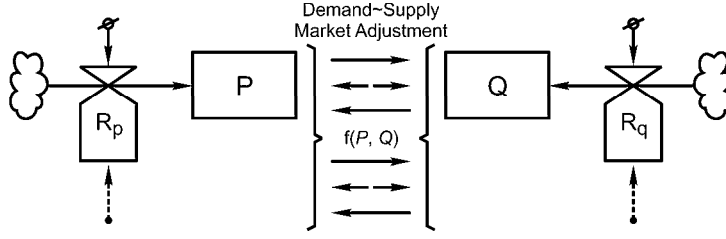
Figure 1: The main elements of the proposed cobweb SD structure

delay and adjustment but somehow it is avoiding to be settled in the common SD model bank of each SD modeler. The main reason for elusiveness of the cobweb model is in it's original form which is not suitable for the straight transformation to the common elements such as Level L and Rate R. The functions of demand $Q_d(k)$ and supply $Q_s(k)$ can be specified in the form:

$$Q_d(k) = a + bP(k) \tag{1}$$

$$Q_s(k) = c + dP(k-1) \tag{2}$$

where a, b, c and d are parameters specific to individual markets. $P(k)$ and $Q_s(k)$ should be restricted to the positive values. In the cobweb model it is assumed that in any one time period producers supply a given amount (determined by the previous time period's price) and then price adjusts so that all the product supplied are bought by customers. If we write this in the form of equation then $Q_d(k) = Q_s(k)$ which enables us to state that the price is:

$$P(k) = \frac{d}{b}P(k-1) + \frac{c-a}{b} \tag{3}$$

Equations 1, 2 and 3 are not quite in the proper form in order to perform the transformation to the SD model. One of the things is the time argument $k-1$. The other is the missing Rate R elements and corresponding Δt. One should expect that the transformation will provide the known equations in the familiar form for structure shown in Fig. 1. The developed model should enable us to examine the properties of the cobweb model and also to consider it's structural and incursive perspective.

As the Wiener's cybernetics [19] principle stands firm in the systems theory the cobweb model principle stands as the basic linearized principle construct for the systems interaction dependance and will probably be further more the basic starting tool for the quantitative analysis of complex systems.

2 Transformation to SD form

Fig. 1 shows, that the price P and quantity Q of the goods should be stated as the Level elements depending on the Rates which determine the change in price and quantity supplied. The theory behind the cobweb model states that quantity supplied in present depends on the ruling price in the past. Therefore the price and the quantity supplied should be dependant variables as illustrated in Fig. 1. Restating the above Equations eliminating the time argument $k-1$ gives us the following set of equations:

$$Q_d(k+1) \quad = \quad a + bP(k+1) \tag{4}$$

$$Q_s(k+1) \quad = \quad c + dP(k) \tag{5}$$

$$P(k+1) \quad = \quad \frac{d}{b}P(k) + \frac{c-a}{b} \tag{6}$$

The Eq. 4, 5 and 6 will enable the determination of the Rates elements. Let us determine the Rate element for the change of Price $R_P(k)$. As the equations are in the difference form the Rate will be determined as $R(k) = L(k+1) - L(k)$:

$$R_P(k) = P(k+1) - P(k) = \frac{c + dP(k) - Q_s(k)}{b} \tag{7}$$

Little bit more work will be needed for the $R_Q(k)$ since special time consideration had to be taken. We will apply the time arguments of $k+1$ and $k+2$ in order to loose the argument $k-1$ which is present in the Eq. 2:

$$R_Q(k) = Q_s(k+2) - Q_s(k+1) = d\frac{c + dP(k^*) - Q_s(k^*)}{b} \tag{8}$$

Since the time argument k^* with consideration of the Eq. 1 and 2 actually represents the past i.e. argument of $k-1$ we should state the equations for $P(k-1)$ and $Q_s(k-1)$. The Eq. 3 will enable us to state $P(k-1)$:

$$P(k-1) = \frac{bP(k) - c + a}{d} \tag{9}$$

$$Q_s(k-1) = a + P(k-1)\,b \tag{10}$$

The Eq. 10 is set by the fact that $Q_d(k) = Q_s(k)$ and Eq. 4. The consideration of the time argument $k-1$ is necessary in order to perform calculation in the model. At each time step the previous values are needed in order to perform the calculation. By putting the $P(k-1) = \frac{bP(k)-c+a}{d}$ in the Eq. 10 we get:

$$Q_s(k-1) = a + \frac{b^2 P(k) - bc + ab}{d} \tag{11}$$

By putting the Eq. 11 and 9 into Eq. 8 we get the simplified form for the rate equation:

$$R_Q(k) = -a + c - (b-d)P(k) \tag{12}$$

As the result of the above algebraic manipulation the cobweb model could be stated in the standard SD form:

$$P(k+1) \quad = \quad P(k) + \Delta t R_P(k) \tag{13}$$

$$R_P(k) \quad = \quad \frac{c + dP(k) - Q_s(k)}{b} \tag{14}$$

$$Q_s(k+1) \quad = \quad Q(k) + \Delta t R_Q(k) \tag{15}$$

$$R_{Q_s}(k) \quad = \quad -a + c - (b-d)P(k) \tag{16}$$
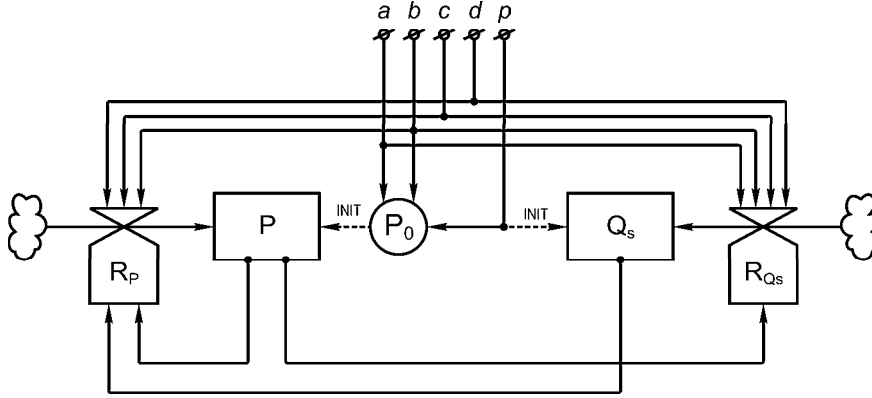
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Figure 2: System Dynamics structure of the cobweb model

with the starting conditions $P(0) = \frac{x-a}{b}$ and $Q_s(0) = x$ where x represents the starting perturbation of the model. In the above set of equations the Δt is introduced which is not present in the classical cobweb model formulation. If $\Delta t = 1$ then the model is equivalent to the classical cobweb.

Fig. 2 shows the SD structure of the cobweb model corresponding to the Eqs. 13, 14, 15 and 16. There are two levels represented, P and Q_s and two rate elements R_P and R_{Q_s}. The model behavior is determined by the input parameters a, b, c and d as well as the perturbation parameter p. The element P_0 represents the initial value of the level element P. The initial value of the level element Q_s is equal to the arbitrary value of perturbation p.

The response of classical cobweb model developed by SD methodology is shown in Fig. 3 and Fig. 4. The applied parameter values with the description of the system response are shown in Table 1.
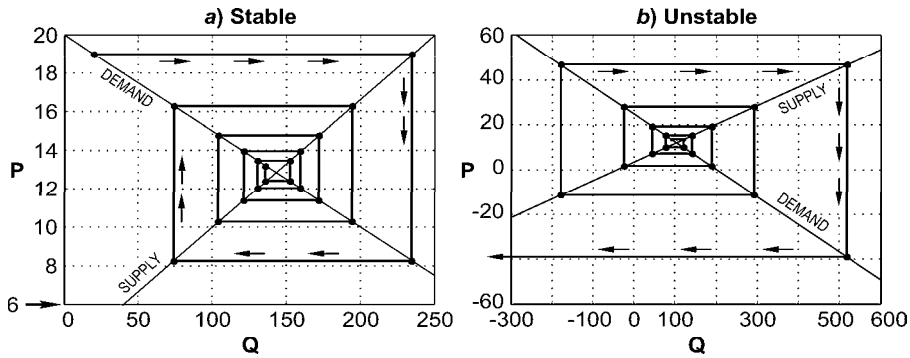

Figure 3: Response of the SD cobweb model: a) Stable, b) Unstable

There are three possibilities: a) Stable system, where the supply and demand converge, b) Unstable system where the supply and demand diverge and c) Dynamically stable shown in Fig. 4, where price and demand neither converge nor diverge.

Dynamically stable response indicates the periodical solution which will be of interest in further examination of the model. In general a solution y_n is periodic if $y_{n+m} = y_n$ for some fixed integer m and all n. The smallest integer for m is called period of the solution. In our case the solution in Fig. 4 is a two-cycle solution.

In general the following definition will be applied [1]:

Definition 2.1 *If a sequence $\{y_t\}$ has e.g. two repeating values y_1 and y_2, then y_1 and y_2 are called period points, and set $\{y_1, y_2\}$ is called a periodic orbit.*

Periodical response of the system is important because real agricultural systems depend on the cyclic behavior and could be controlled only by regarding the periodicity of such systems. Examples from real cases could easily be found in crops as well as in the stock.
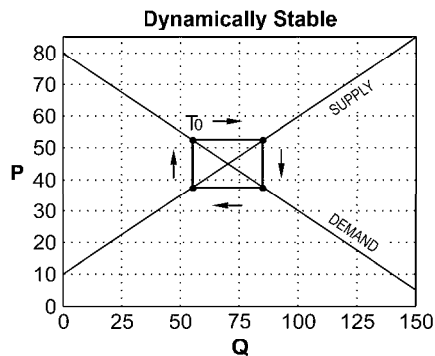

Figure 4: Response of the SD cobweb model: Dynamically stable

2.1 Separation of the structure elements

The structure of the model in Fig. 2 shows that the Price and Quantity are related. However the structure can be represented in a different way. By transforming the cobweb model to SD form the model could become non-autonomous depending on the variable Δt. The following two equations represent the different formulation of the cobweb model:

$$Q_s(k+1) = c + d\frac{Q_s(k) - a}{b} \tag{17}$$

$$P(k+1) = \frac{c + dP(k) - a}{b} \tag{18}$$

This reformulation represents Q_s and P as the non-related quantities. The only bound that exists are the coefficients. In order to formulate the complete SD model the rate elements which determine initial conditions should be determined:

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Table 1: Parameter values

a	b	c	d	p	Response
400	-20	-50	15	20	Stable
200	-8.1	-43	12	90	Unstable
160	-2	-20	2	55	Dyn. Stable

</div>

$$R_P(k) = P(k+2) - P(k+1) \quad = \quad \frac{c + dP(k+1) - a}{b} - \frac{c + dP(k) - a}{b} =$$
$$= \quad \frac{d}{b}(P(k+1) - P(k)) \tag{19}$$

$$R_{Q_s}(k) = Q_s(k+2) - Q_s(k+1) \quad = \quad c + d\frac{Q_s(k+1) - a}{b} - c - d\frac{Q_s(k) - a}{b} =$$
$$= \quad \frac{d}{b}(Q_s(k+1) - Q_s(k)) \tag{20}$$

In order to meet initial conditions of the model the $Q_s(k-1)$ should be determined:

$$Q_s(k-1) = a + \frac{b}{d}(Q(k) - c) \tag{21}$$

Equations for P and Q_s in standard SD form are the following:

$$P(k+1) \quad = \quad P(k) + \Delta t R_P(k) \tag{22}$$
$$R_P(k) \quad = \quad \frac{d}{b}\left(P(k) - \frac{bP(k) - c + a}{d}\right) \tag{23}$$
$$Q_s(k+1) \quad = \quad Q_s(k) + \Delta t R_{Q_s}(k) \tag{24}$$
$$R_{Q_s}(k) \quad = \quad \frac{d}{b}\left(Q_s(k) - \left(a + \frac{b}{d}(Q_s(k) - c)\right)\right) \tag{25}$$

Eqs. 22, 23, 24 and 25 represents the cobweb model in SD form separated as shown in the Fig. 5. Note that the terms for P and Q_s are related only to the coefficients a, b, c, d and p. $P(k+1)$ is dependent only on the value of $P(k)$ and coefficients a, b, c, d and p, not on the Q_s. Respectively for the $Q_s(k+1)$.
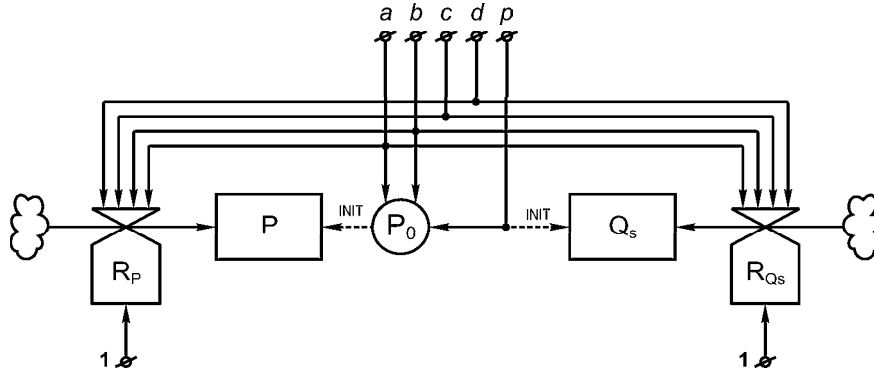

Figure 5: Cobweb model in SD form $\sim$ separated elements

2.2 Anticipative formulation

Comparison of the structures shown in Fig. 2 and Fig. 5 indicates, that P and Q_s depend only on the parameter values a, b, c, d and p i.e. the initial conditions. Eqs. 22, 23, 24 and 25 enable the determination of entire anticipative (future event) chain while equation:

$$P(k-1) = \frac{bP(k) - c + a}{d} \tag{26}$$

and Eq. 21 enable the determination of feedback (past event) chain. The representation of the Feedback $\sim$ Anticipative chain is shown in Fig. 6. The dynamics of interest is therefore the chains dynamics which is dependant on the parameters $a(t), b(t), c(t), d(t)$ and $p(t)$. Both chains are actually dependant on strategy dynamics which could be formulated as the $f(a, b, c, d, p, t)$.
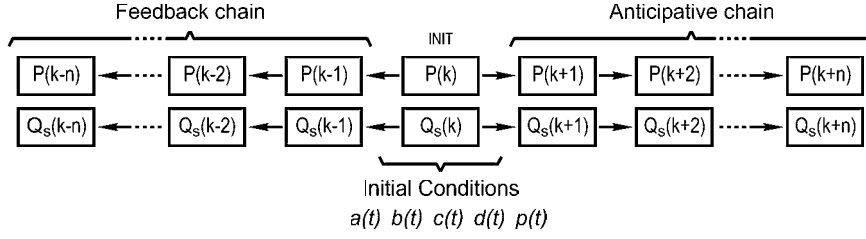

Figure 6: Feedback $\sim$ Anticipative chain

Application of hyperincursive algorithm and inspection of gained equations with Dubois' [6] formulation of logistic growth and previous research [11, 12] yields the following set of equations for the hyperincursive cobweb model:

$$P(k+2) = \frac{d}{b}\left(A - \left(\frac{bB - c + a}{d}\right)\right) \tag{27}$$

$$Q_s(k+2) = \frac{d}{b}\left(C - a - \frac{b}{d}\left(D - c\right)\right) \tag{28}$$

with initial conditions:

$$P_0(k+1) = \frac{p - a}{b} \tag{29}$$

$$P_0(k) = \frac{bP_0(k+1) + a - c}{d} \tag{30}$$

$$Q_{s0}(k+1) = p \tag{31}$$

$$Q_{s0}(k) = a + \frac{b}{d}\left(Q_{s0}(k+1) - c\right) \tag{32}$$

Terms $A(k)$ and $B(k)$ in Eq. 27 could be replaced by the terms $P(k+1)$ or $P(k)$, similarly $C(k)$ and $D(k)$ in Eq. 28 by $Q_s(k+1)$ or $Q_s(k)$. This yields 16 different combinations of system defined by Eq. 27 and Eq. 28 that should be studied. The appropriate explanation of modified system structure should follow the techniques of graphical solutions for homogenous difference equations [22].

The system combination further examined will have the following terms: $A = P(k+1), B = P(k), C = Q_s(k+1)$ and $D = Q_s(k)$. This yields the following set of equations:

$$P(k+2) \quad = \quad \frac{d}{b}\left(P(k+1) - \left(\frac{bP(k) - c + a}{d}\right)\right) \tag{33}$$

$$Q_s(k+2) \quad = \quad \frac{d}{b}\left(Q_s(k+1) - a - \frac{b}{d}\left(Q_s(k) - c\right)\right) \tag{34}$$

Eq. 33 and Eq. 34 with initial conditions stated by Eqs. 29 $\sim$ 32 could be modelled as shown by the structure in the Fig. 7. The structure represents the cobweb model in hyperincursive form modelled by classic SD elements. The Euler integration method is applied with time-step $\Delta t = 1$.
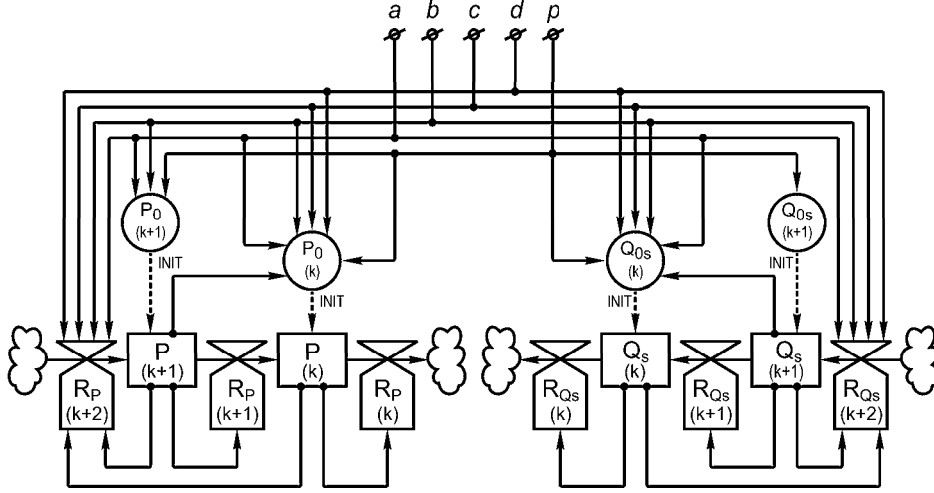

Figure 7: Structure of the hyperincursive Cobweb model; Euler integration, $\Delta t = 1$

Eq. 33 and Eq. 34 could be reformulated in order to show the dependency of the future-present-past events:

$$P(k) \quad = \quad \frac{bP(k-1) + a - c}{d} + \frac{b}{d}P(k+1) \tag{35}$$

$$Q_s(k) \quad = \quad \frac{b}{d}Q_s(k+1) + \frac{b}{d}Q_s(k-1) + a - \frac{bc}{d} \tag{36}$$

Eq. 35 and Eq. 36 state that the value of the present is dependent on the past as well as on the future. This paradoxical statement is realizable since the formulation of feedback anticipative chain could be stated. Fig. 7 has two delay chains, one for P and one for Q_s. One might notice, that the level and rate elements are dependant only on the coefficients and initialization values.

3   Results

Fig. 8 represents some examples of hyperincursive cobweb model. The matrix of graphs in Fig. 8 represents different modes of cyclic behavior named by the shape of the 2d mapping of $P(k+1), P(k+2)$ (second column represents the shapes in 2d mapping). Each row is named by second column 2d mapping shape and represents the response of the model

in four different mappings in each column different mapping. First column represents the dynamics in the classical cobweb form. 3rd and 4th columns represents the 3d mapping which is 2d mapping expanded with the time step k on x-axis. Fig. 9 represents cobweb and 2d mapping of the system before, in and after pentagon synchronization. Here pentagon shape is taken only as the example. Each of the presented shapes in Fig. 8 could be observed in the similar way. First pair of pictures in first row represents the behavior of the system before the synchronization. Notice the arrows in the first row under 2d mapping which show the movement of the vertices of the graph. The vertices converge to the one point which is shown in the second row. Points cover one another and in the case of pentagon synchronization estimated values of synchronization parameters are: $a = 400$, $b = -20$, $c = -50$, $d = -12.3671$ and $p = 160$. Third row in Fig. 9 represents the state of the system after the pentagon synchronization where arrows represent the divergent movement of the vertices points of the graph. The system is in transition to the next full synchronization which is estimated as the quad synchronization where parameter d is near 0. The synchronization representation in *3d* might be better observed than in the *2d* mapping. The responses of the system in Fig. 8 were gained according to the parameter values gathered in Table 2. The change was made in parameter d which yielded the synchronization patterns as shown by the shape column. The parameter values were gained by the simulation where the range of parameter d was set as of $[-40, 40]$ with $\Delta d = 0.001$. The condition for parameter values determination was set by the rule of acceptable error between simulation steps and definition 2.1 of synchronization .

4 Periodicity of the system

The z-transform is the basis of an effective method for solution of linear constant-coefficient difference equations. It essentially automates the process of determining the coefficients of the various geometric sequences that comprise a solution [4]. The application of z-transform on the Eq. 33 and Eq. 34 with initial conditions stated by Eqs. 29 $\sim$ 32 gives:

$$Y(z) = \frac{-y_1 z + y_0 dz - y_0 z^2}{-1 + dz - z^2} \tag{37}$$

Inverse z-transform yields the following solution:

$$Y^{-1}(z) = 2^{-1-n} y_0 \left(d - \sqrt{-4 + d^2}\right)^n - \frac{y_1 \left(d - \sqrt{-4 + d^2}\right)^n}{2^n \sqrt{-4 + d^2}} +$$

$$+ \frac{2^{-1-n} y_0 d \left(d - \sqrt{-4 + d^2}\right)^n}{\sqrt{-4 + d^2}} + 2^{-1-n} y_0 \left(d + \sqrt{-4 + d^2}\right)^n +$$

$$+ \frac{y_1 \left(d + \sqrt{-4 + d^2}\right)^n}{2^n \sqrt{-4 + d^2}} - \frac{2^{-1-n} y_0 d \left(d + \sqrt{-4 + d^2}\right)^n}{\sqrt{-4 + d^2}} \tag{38}$$
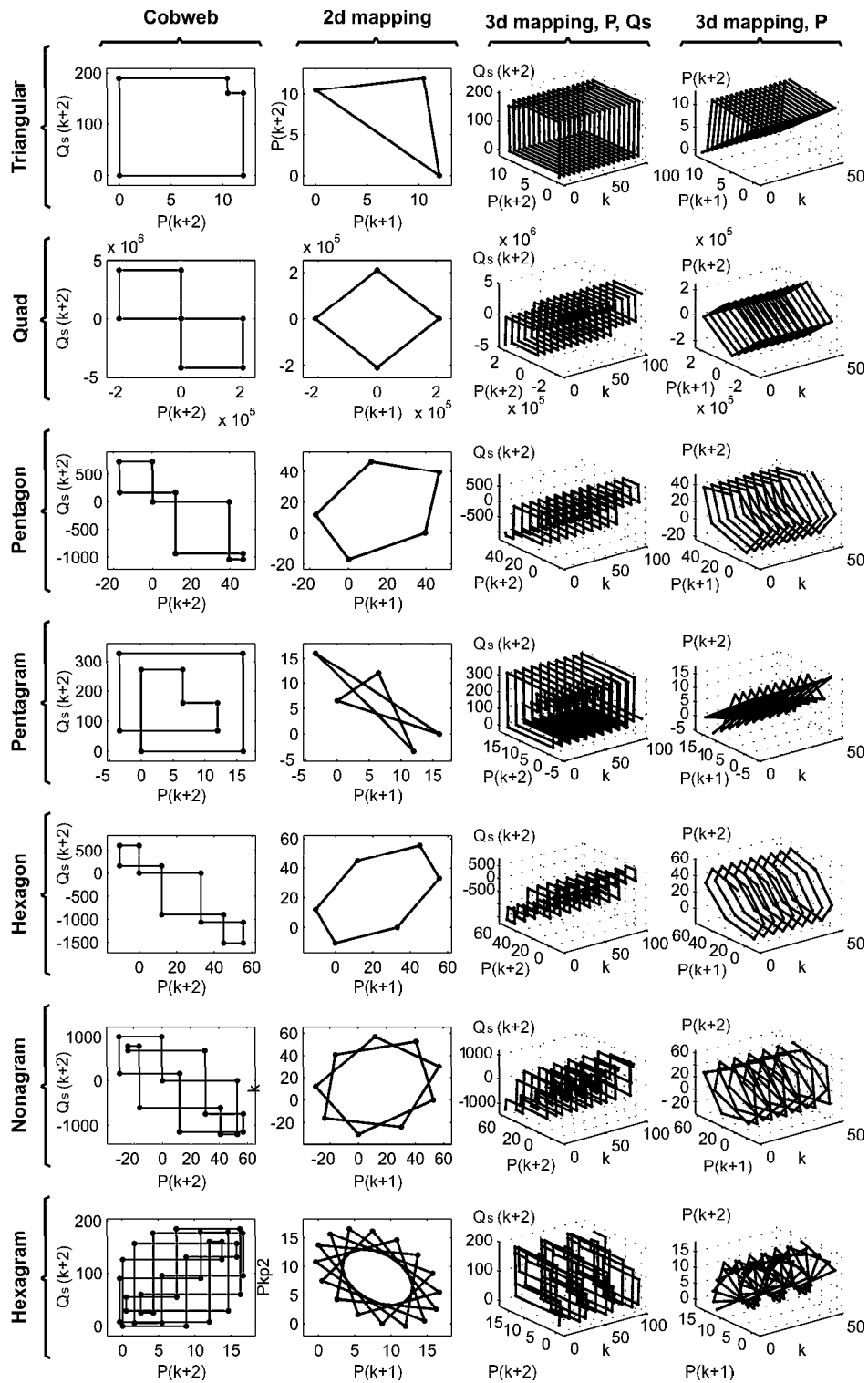
Figure 8: Results of the hyperincursive cobweb model; examples of cyclic behavior; notice rows and columns naming
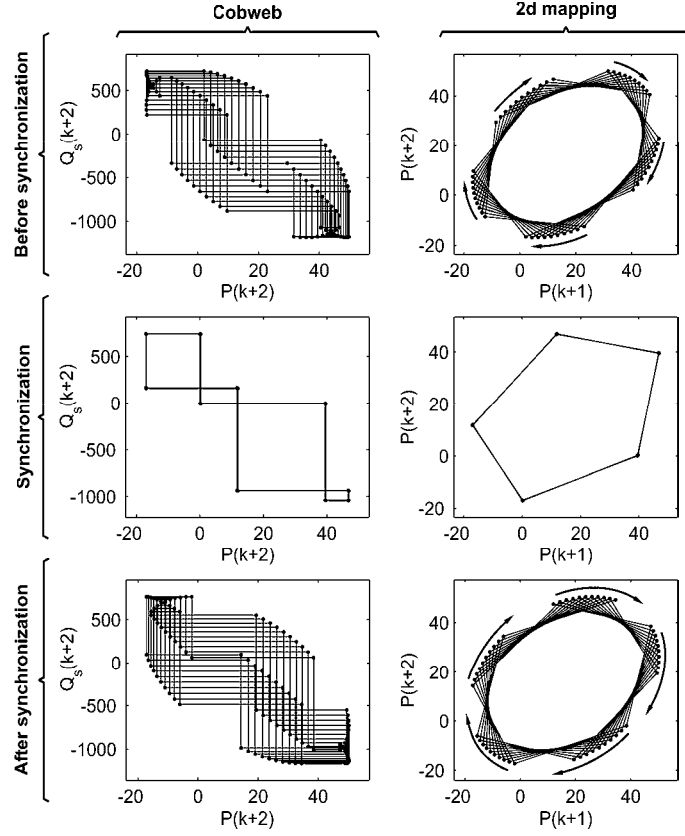
Cobweb   **2d mapping**

Figure 9: Cobweb and 2d mapping of the system before, in and after pentagon synchronization

In order to gain conditions for the periodic response of the system the following equation should be solved:

$$Y^{-1}(z) \quad = \quad y_0 \tag{39}$$

Let us compute a numerical example of periodic solution applying the z-transform. The period examined will be the period of 9 i.e. $n = 9$. In Eq. 39 one should put the condition $n = 9$. One of the possible solutions for the initial condition worth of examination is the following:

$$d \quad = \quad \frac{1}{\left(\frac{1}{2}(-1 + i\sqrt{3})\right)^{\frac{1}{3}}} + \left(\frac{1}{2}(-1 + i\sqrt{3})\right)^{\frac{1}{3}} \tag{40}$$

The term $(-1 + i\sqrt{3})^{\frac{1}{3}}$ (let us denote the term as z^*) could be expressed by three different imaginary values in polar form:

$$z_1^* \quad = \quad \sqrt[3]{2}\left(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}\right) \tag{41}$$

$$z_2^* \quad = \quad \sqrt[3]{2}\left(\cos\frac{8\pi}{9} + i\sin\frac{8\pi}{9}\right) \tag{42}$$

$$z_3^* \quad = \quad \sqrt[3]{2}\left(\cos\frac{14\pi}{9} + i\sin\frac{14\pi}{9}\right) \tag{43}$$

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By putting Eq. 41, Eq. 42 and Eq. 43 into Eq. 40 and performing trigonometric reduction one gets the following solutions:

$$d_1 = 2\cos\frac{2\pi}{9} \quad d_2 = 2\cos\frac{4\pi}{9} \quad d_3 = 2\cos\frac{8\pi}{9} \tag{44}$$

By inspecting the Eq. 40 and considering the equation for the roots of complex numbers [15]:

$$\sqrt[n]{z} = \sqrt[n]{r}\Big(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\Big) \tag{45}$$

the general form of the solution for the parameter d could be assumed:

$$d = 2\cos\frac{2\pi m}{n} \tag{46}$$

where n is the period and $m = 1, 2, 3, ..., n-1$. Similar procedure could be performed for the arbitrary period n. More general solutions which regards also the parameter b which was fixed for the purpose of determination of solutions is:

$$d = 2b\cos\frac{2\pi m}{n} \tag{47}$$

The solutions could in some cases be expressed in alternative algebraic or trigonometric form. Tab. 2 shows some of the solutions for the parameter d. Alternative solutions could be expressed as the roots of the polynomial. The table incorporates the Shape symbols which are important at the study of the response of dynamical systems. Especially this is the case at the examination of complex nonlinear dynamical systems [16] or their representation in SD form [29]. Mappings of the system and visualization of the periodic solution is therefore important for the analyzing of periodic or chaotic solutions of differential and discrete difference equations. Shape description is taken from the vocabulary of proper shapes although the response of the system is primarily not proper. Numerical values of the solutions for parameter d are important since this values also confirm the findings of Sonis [24, 5] about the domain of attraction for 2D dynamics by n-dimensional linear bifurcation analysis. One of the important conditions gained by the proposed inspection is the value of the period $n = 10$ which is in close relation to the period $n = 5$. The value of parameter d is $d = \frac{1}{2}(1 + \sqrt{5})$ with numerical value 1.61803... This solution represents the "Golden Ratio" (ϕ). Some of the different representations of solution for parameter d value at period $n = 10$ are:

$$d_{10} = \phi = 2\cos\frac{\pi}{5} = \frac{1}{2}(1 + \sqrt{5}) = 1.61803398874989484820... \tag{48}$$

The first solution of parameter d at period $n = 10$ connects the considered discrete system with the Fibonacci numbers given by the infinite series:

$$d_{10} = \phi = 1 + \sum_{u=1}^{\infty} \frac{(-1)^{u+1}}{F_u F_{u+1}} \tag{49}$$

The fact, that the periodicity conditions of the examined discrete system incorporates the golden ratio number ϕ could be observed in other studies [8] of complex nonlinear

expansions of the basic cob-web systems e.g. Brock and Hommes "Almost Homoclinic Tangency Lemma". One should expect that the symmetric response in $n - mapping$ should follow the pattern with the match in certain point of solution with the ϕ condition. The source of the mentioned condition is presented by the preceding procedure. (The value of parameter d for mentioned period $n = 5$ is $d = \frac{\sqrt{5}-1}{2} = 0.61803...$ often called the "Golden Mean".) (*)Since the period 2 is on the boundary of the solution the periodic response of the system depends on the initial conditions. Example of numerical values of the period-2 response: $a = 1, b = 1, c = -1, d = -2, p = 1$. (**) The value for tetragon is taken in the limit since the system of equation returns the undefined value when $d = 0$ therefore one should consider the tetragon period condition as the value approaching to zero i.e. $d \to 0$. In this case the system response is undetermined ($\frac{0}{0}$) in it's critical point.

Table 2: Synchronization parameter values

Shape	Description	a	b	c	d	p
△	Triangular	400	−20	−50	20.0000	160
◇	Quad	400	−20	−50	−0.0010	160
⬠	Pentagon	400	−20	−50	−12.3671	160
✫	Pentagram	400	−20	−50	32.3620	160
⬡	Hexagon	400	−20	−50	−20.0000	160
✪	Nonagram	400	−20	−50	−6.9450	160
✸	Hexagram	400	−20	−50	15.3070	160

Discrete $2d - map$ should be analyzed according to the variation of parameter d and the determinant $\Delta = p^2 - 4q$. The range of the cyclical behavior is determined by the classical imaginary solution of the dynamical system which is in our case defined by the characteristic equation:

$$\lambda = \frac{-2b + d \pm \sqrt{-4b^2 + d^2}}{2b} \qquad (50)$$

Stability result corresponds to the polynomial $\lambda^2 = trJ\lambda - detJ$ where periodic solutions will be considered. One should consider e.g. [5] for details and research in the field of discrete dynamics cobweb models [28, 13]. The ordering of the equilibriums is determined by the general Eq. 47. The rational fraction $\frac{m}{n}$, which is in our case transformed by the Eq. 47 to the value of the parameter d, corresponds to the Farey sequence which could be represented by the Farey tree. Important to notice is the sequences of Fibonacci numbers [17] $\{F_i/F_{i+1}, i \in \aleph\}$ and $\{F_i/F_{i+2}, i \in \aleph\}$. Since the observed system is linear the change in parameter b will not influence the periodicity condition. The emergence of the system periodic stability in the shape of n-sided polygon could be observed not only in economical systems [21, 20]; the n-sided polygon and the Farey tree organization of the equilibria could be observed in the technical systems as for example in laser control as the paradigm of the chaotic system [23].

4.1 Bifurcation Analysis - Extension to Nonlinear systems

Periodicity conditions in previous section are somehow general and could be transferred from linear systems to nonlinear, see for example [5, 24]. In order to observe the pseudo-bifurcation response of the initial 2-d discrete map the bifurcation was performed for the change in parameter d in the range $d \in [-2, 2]$ which covers the whole periodicity area of the studied model. Fig. 10 represents pseudo bifurcation diagram for the linear system. One could observe the strong periodicity points which are marked with the vertical lines and correspond to the previously studied Farey tree sequence of polygons. X-axis represents the variation in the parameter d and on the Y-axis the values of P_k are shown. The graph shows the periodicity response as major gaps in the bifurcation with the pole at the origin of X-axis.
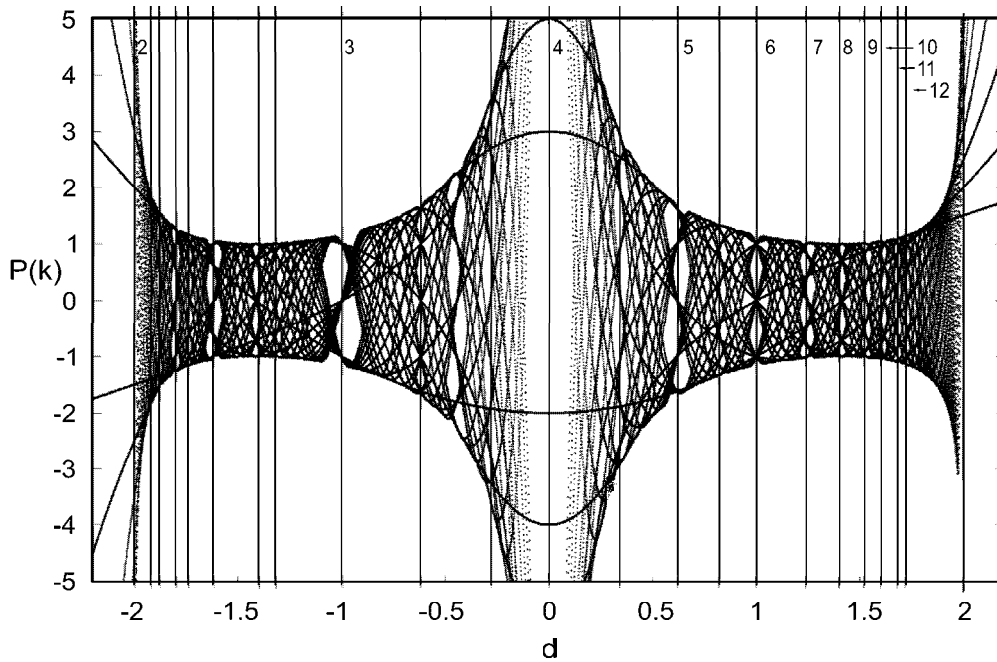

Figure 10: Bifurcation diagram - initial linear 2d discrete map

The observation developed so far leads to the following proposition:

Proposition 4.1 *Periodicity conditions in the linear 2-d map of the cobweb anticipative system exist in the nonlinear expansion of the system.*

If we consider the proposed socio-economic model by introducing the nonlinearity there are several implications [27, 30, 31]: a) neccessity of the nonlinear equation application in order to simulate the system, b) evolutionary character of the socio-economic systems could only be revealed via computer simulation and c) nonexistence of analytical solutions. The obvious approach to model complex dynamical systems is therefore the mixture of several approaches, continuous simulation, discrete-event simulation and application of nonlinear dynamics models. Important fact that should be considered at the modeling of complex systems is the impossibility of prediction [27] as the immanent characteristic which is present in nonlinear chaotic model representations. Consider generic alteration

of the initial anticipative cobweb model:

$$
\begin{aligned}
P_K(k+1) &= P_K + P_{KP1}(k) - \\
&- \left(P_K(k) + \frac{1}{P_Z(k)P_K(k)} \right) \\
P_{KP1}(k+1) &= P_{KP2}(k) \\
P_{KP2}(k+1) &= \frac{d}{b}\left(P_{KP1} - \frac{bP_K(k) - c + a}{d} \right) \\
P_Z(k+1) &= P_Z(k) + P_K(k)P_{KP1}(k) - \\
&- vP_Z(k)
\end{aligned}
\tag{51}
$$

Slight modification of initial Hicks' model [22] with applied accelerator principle [34] gives the interesting response. The system can be represented in three dimensions which reveals the periodicity of the system for which the previously determined conditions of Farey tree generally still holds. Fig. 11 shows the *3d* bifurcation diagram for the altered model. One can see the four attractors which are simultaneous and represent the four possible equilibrium states for the trade dynamics. The 4-cycle characteristic is preserved at the alteration of the parameter v which could be observed in the Fig. 12. The four dots on the center-right side of the figure represents the four-cycle characteristic of the response. The larger orbits indicate the steep change in the modus of the system. However the analytical proof of the periodicity would be hard. The underlying Farey sequence define the adapted nonlinear *2-d* discrete map. Such evidences are also found in other works in nonlinear system analysis for example [8] or in the recent works of Swedish economist T. Puu.
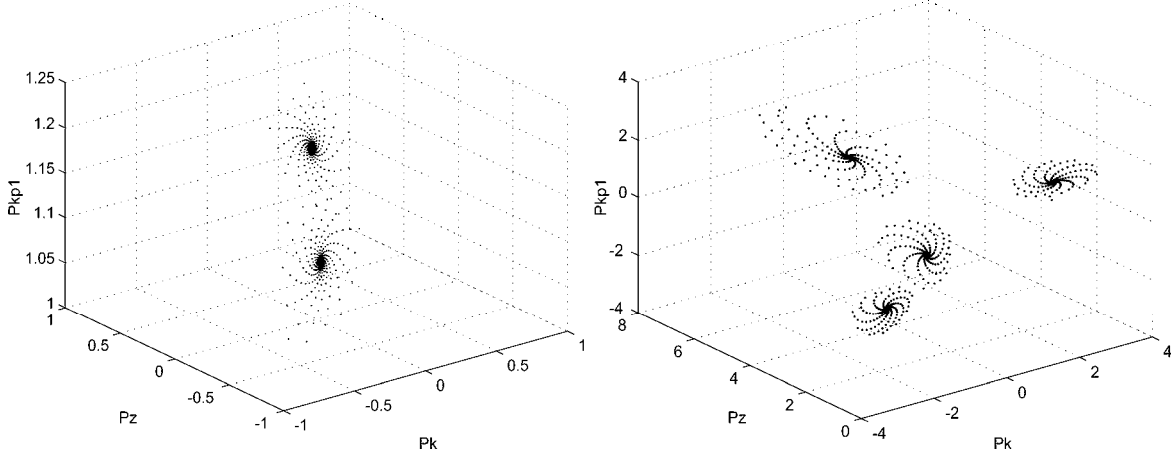

Figure 11: *left)* Emergence of two Synchronous Attractors in the nonlinear case where $d = 0.3833$ and $b_1 = 0.33$ and *right)* Emergence of Four Synchronous Attractors where $d = 0.160793$ and $b_1 = 0.18$

In order to analyze the preservation of the periodic solutions the most significant periodic solution i.e. the period 6 has been applied to the system Eq. 51 which is restated in the following form:

$$
\begin{aligned}
P_{KP1} &= \frac{d}{b}P_{KP1} - P_K + c - a \\
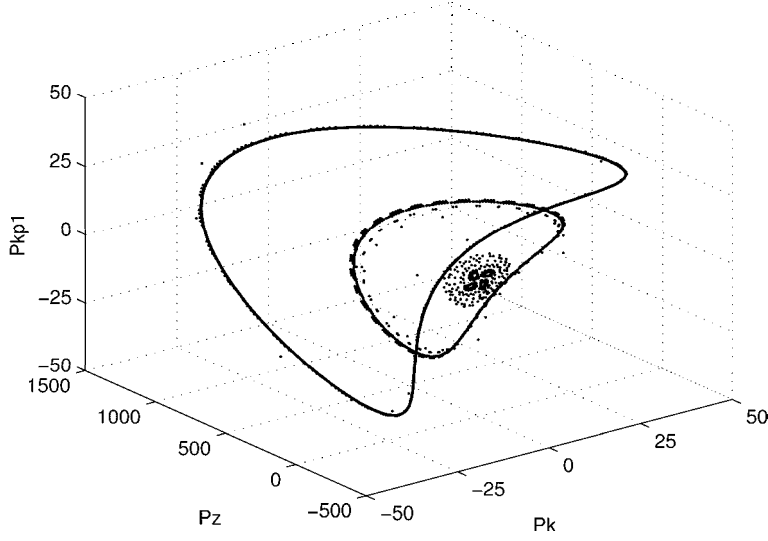P_K &= P_{KP1} - \frac{1}{P_Z P_K}
\end{aligned}
$$

Figure 12: Preservation of Four Synchronous Attractors in the nonlinear case where $d = 0.16404706$ and $b_1 = 0.18$

$$P_Z \;=\; P_Z(1-v) + P_{KP1}P_K \tag{52}$$

The following condition has been applied to the Eq. 52:

$$-P_{KP1} - \frac{d\,P_{KP1}}{b} + \frac{d\left(-a + c + \frac{d\,P_{KP1}}{b} - P_K\right)}{b} +$$

$$P_K + \frac{1}{P_K\,P_Z} = 0$$

$$-a + c - P_{KP1} + \frac{d\,P_{KP1}}{b} - P_K + \frac{1}{P_K\,P_Z} -$$

$$\frac{1}{\left(P_{KP1} - \frac{1}{P_K\,P_Z}\right)\left(P_{KP1}\,P_K + (1-v)\,P_Z\right)} = 0$$

$$-\left(P_{KP1}\,P_K\right) + \left(-a + c + \frac{d\,P_{KP1}}{b} - P_K\right)$$

$$\left(P_{KP1} - \frac{1}{P_K\,P_Z}\right) - (1-v)\,P_Z +$$

$$(1-v)\left(P_{KP1}\,P_K + (1-v)\,P_Z\right) = 0$$

which yields the periodicity solutions that are related to the system attractor. For example, for period 10 the initial values are: $P_{KP1} = -1$, $P_K = -1.61803$, $P_Z = 1.61803$, $d = 1.618033989$, $v = 1$, for period 7: $d = \cos\frac{2\pi}{7}, v = 1, P_{KP1} = -2.16007$, $P_K = -0.533493$, $P_Z = 1.15238$.

For the nonlinear mapping symmetry could be expected as shown with numerical examination of the system for the parameter $d \in [0.3833, 0.4538]$

As the illustration of the presented concept and it's relation to the real world [39] the following example will be considered: Fig. 14 shows the real world data for the Number of births, daily, Quebec, January 1, 1977 to December 31, 1990 [25]. Other series closer to
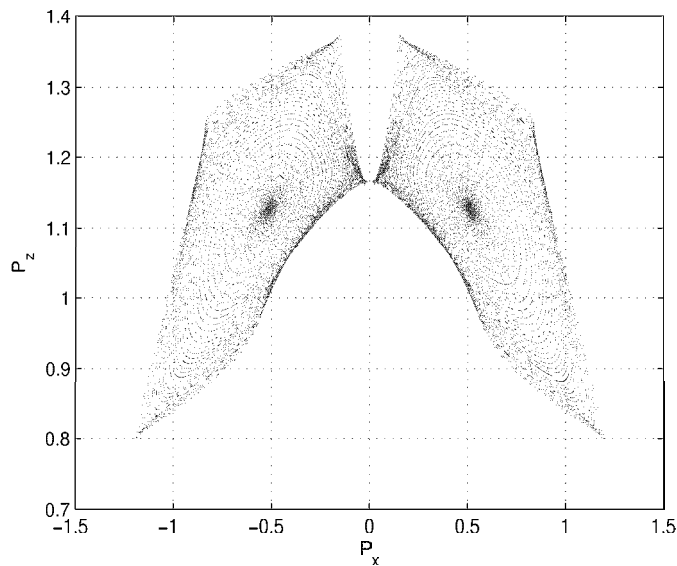
Figure 13: Symmetry of the nonlinear map for $d \in [0.3833, 0.4538]$

the Supply$\sim$Demand system could also be considered here. This time series shows nothing in particular. However, if we consider the data by the Poincaré first-return map [35] as shown in Fig.15 one could observe the strong quad ($\Diamond$) polygon showing the basic periodic characteristic of considered system.
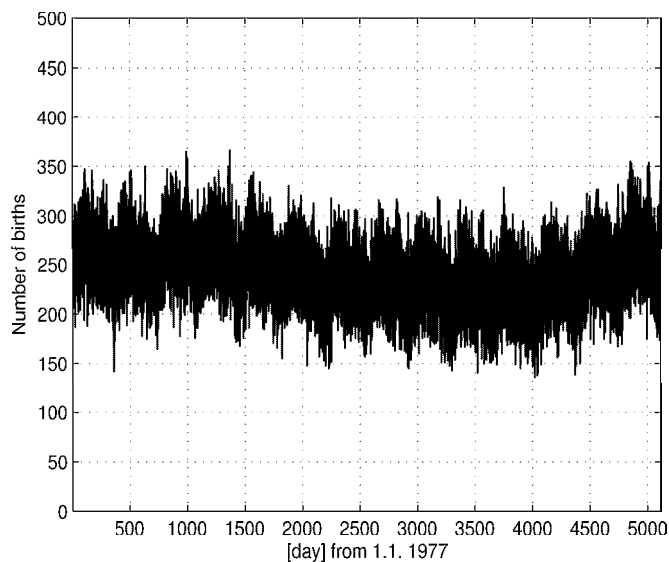

Figure 14: Real world system response - number of births from 1.1.1977

The periodically driven system is identified in the Fig.15 where the strong 4-cycle resonance is shown. One of the common periodicity conditions is 6-cycle as the cycle of possible optimal response.

If we consider the proposed nonlinear system by the examination of Lyapunov expo-
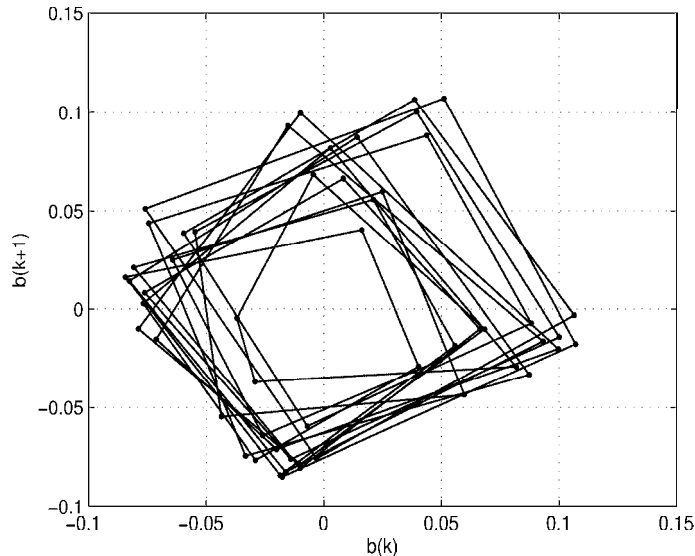
Figure 15: Poincaré first-return map

nents defined by Eq. 53 the chaotic property is indicated [36].

$$\frac{1}{n} \left(\ln |f'(x_0)| + \ln |f'(x_1)| + \ldots + \ln |f'(x_{n-1})| \right) \tag{53}$$

Eq. 53 tells us that the Lypunov exponent is the rate of divergence of the two trajectories. Stated differently, we have here the average of the natural logarithm of the absolute value of the derivatives of the map function evaluated at the trajectory points. If the application of the map function to two nearby points leads to two points further apart then the absolute value of the derivative of the map function is greater than 1 when evaluated at those trajectory points. If the absolute value is greater than 1 then the logarithm is *positive*. If the trajectory points continue to diverge on the average, then the average of the logarithm of the derivatives absolute values is positive. In our case the *Gram-Schmidt* orthogonalization procedure [37, 38, 18] which gives the following results for the three Lyapunov exponents, λ_1, λ_2 and λ_3 shown in Fig.16.

From the Fig.16 the stable regions where all three the Lyapunov exponents λ_1, λ_2 and λ_3 are less than 0 could be observed. For the studied system the regions of stability are important. As the example of the application of Lyapunov exponents one should consider parameter bifurcation in the parameter space of interest in order to identify regions of stability. As an example, the computation of the Lyapunov exponents for parameters $d \in [0.1, 4]$ and $v \in [0.5, 0.6]$ was performed indicating the region of stability shown in the Fig. 17. The blue region shows the chaotic parameter space of the system while the white region indicates the stable region. Such stability regions are usually periodic regions obeying previously mentioned Farey sequence rule. At the analysis of nonlinear system the regions of stability are of particular interest [33, 32, 10, 9, 14]. This regions are dependant on the parameter values and initial conditions; here we have the function $f(A, X_0, Y_0, Z_0)$ where A represents set of parameter values, X_0, Y_0 and Z_0 represents the set of initial values. Analytical determination of boundary conditions of periodicity regions is difficult task and
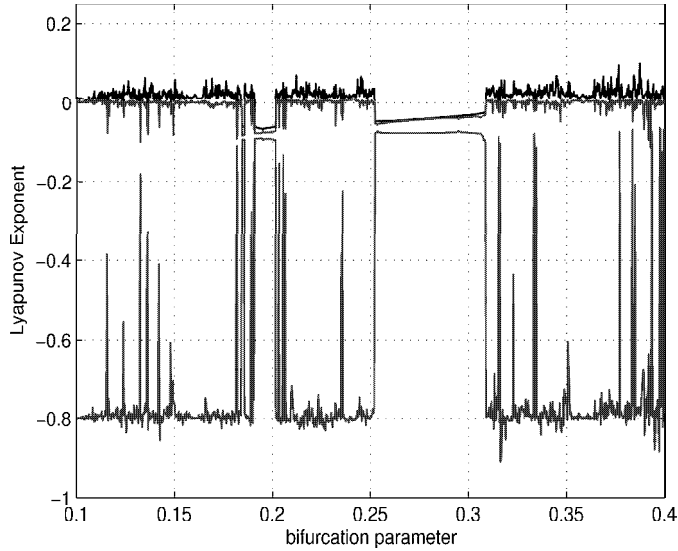
Figure 16: Lyapunov exponents for 3-d discrete nonlinear map for λ_1, λ_2 and λ_3

in many cases impossible. For the determination of the stability regions, the parameter space search is performed as the aggregate Lyapunov estimator dependant on the system response for the function $f(A, X_0, Y_0, Z_0)$. However, one should be aware, that a full understanding of a nonlinear system might require computing solutions for essentially all initial conditions [26]. Since this is infeasible, we would like some simple ways to summarize the possible behaviors as proposed by the Farey tree sequence. Here the application of powerful parallel computing provides the basic platform for systematic [40, 41] nonlinear system analysis. But one should be warned that only the analytical explanations of the nonlinear systems' behavior provide methodologically correct characterization.

4.2 Equilibrium Analysis

Equilibrium condition for the P segment of the hyperincursive cobweb system could be stated as:

$$\frac{d}{b}\left(\frac{p-a}{b} - \frac{\frac{p-c}{d} - c + a}{d}\right) = \frac{p-a}{b} = \frac{p-c}{d} \tag{54}$$

The equilibrium values of the parameters for the P segment of the system are: $a = c = p$ and $b = d$.

Equilibrium condition for the Q_s segment of the hyperincursive cobweb model could be stated as:

$$\frac{d}{b}\left(p - a - \frac{b}{d}\left(a + \frac{b}{d}(p - c) - c\right)\right) = p = a + \frac{b}{d}(p - c) \tag{55}$$

Q_s segment of the system has no solution for the equilibrium values of the parameters. When the equilibrium conditions for the P segment of the system are considered in all the
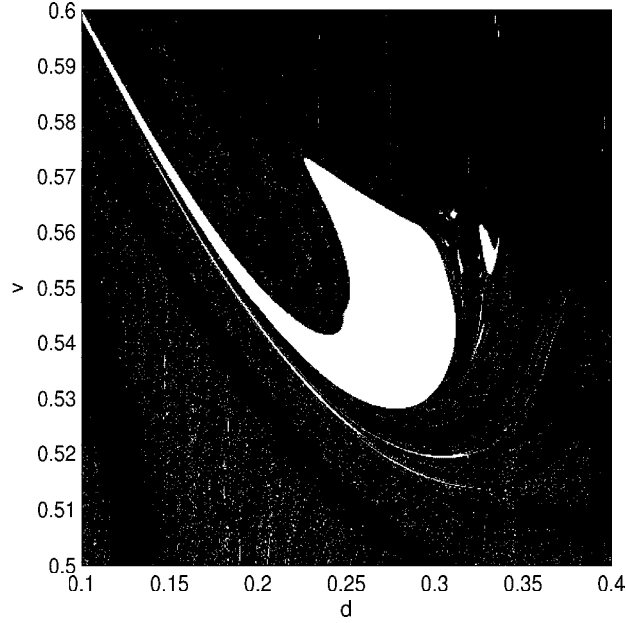
Figure 17: Lyapunov exponents stability region (white) for 3-d discrete nonlinear map for parameter space $d \in [0.1, 4]$ and $v \in [0.5, 0.6]$

cases, the Q_s segment of the system could not be in stable state. Graphical presentation of the equilibrium conditions $a = c = p$ and $b = d$ is shown in Fig. 18. The hexagon shape is known as the possible optimal shape in the context of spatial economics [21]. The existence of hexagonal shapes is known - in the space economy and explained by the structural stability [21].
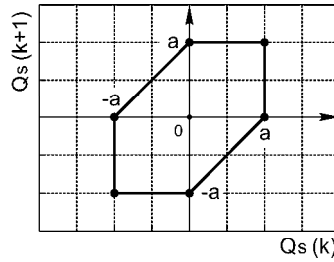

Figure 18: Response of the Q_s segment of the system while in equilibrium

Proposition 4.2 *Equilibrium condition for the P segment of the anticipative cobweb system defined by the equations from Eq. 29 to Eq. 34 is: $a = c = p$ and $b = d$. At this conditions, the Q_s segment of the system has the response of hexagon shape with vertices $\{(a, 0), (a, a), (0, a), (-a, 0), (-a, -a), (0, -a)\}$ in $Q_s(k)$, $Q_s(k + 1)$ mapping.*

While the response of the system for the P segment is in equilibrium, the Q_s segment of the system has the hexagon-like shape with significant dimension of parameter a value and edges dimension of a and $a\sqrt{2}$.

20

Proposition 4.3 *Triangular ($\triangle$) i.e. three-period response in 2-dimensional mapping is determined by the condition $b = -d$.*

In order to gain the term for the period in the P values one should apply Prop. 2.1. The values for times $1, 2, 3, 4$ should be symbolically expressed. By inserting the Eq. 29 and Eq. 30 into Eq. 33 the following term is gained:

$$P(k+2) = \frac{d}{b}\left(\frac{p-a}{b} - \frac{a - c + \frac{b(p-c)}{d}}{d}\right) \tag{56}$$

By repetition of similar procedure the equation for $P(k + 3)$ considering the period 4 condition i.e.: $P(k + 3) = P(k)$ one should get the following equation:

$$\frac{d}{b}\left(\frac{d}{b}\left(\frac{p-a}{b} - \frac{a - c + \frac{b(p-c)}{d}}{d}\right) - \frac{p-c}{d}\right) - \frac{p-c}{d} = 0 \tag{57}$$

with solution $b = -d \in \Re/\{0\}$.

Proposition 4.4 *Hexagon ($\bigcirc$) i.e. six-period response in 2-dimensional mapping is determined by the condition $b = d$.*

The symbolic solution is gained by Def.2.1. The values for times $1, \ldots, 6$ should be symbolically expressed. Determination of $P(k + 2)$ is based on expressions for $P(k + 1)$ and $P(k)$, determination of $P(k + 3)$ is based on expressions for $P(k + 2)$ and $P(k + 1)$ etc. Periodic condition is expressed as $P(k + 6) - P(k) = 0$ with solution $b = d \in \Re/\{0\}$.

The general periodicity solution has been determined by the *z-transform* analysis. For parameter d the solution is:

$$d = 2\cos\frac{2\pi m}{n} \tag{58}$$

where n is the period and $m = 1, 2, 3, ..., n - 1$. Here, as mentioned, the above procedure could be performed for the arbitrary period n.

5 Discussion

The story revealed from the developed hyperincursive model raises the following questions: a) Does a change in the strategy changes the structure or does it change only the relations between the elements of the structure? b) Does changing of the strategy change the future as well as the past?
A change in the strategy would mean a new and different future and should also mean a different past if the change in the strategy would occur earlier. The hyperincursive cobweb model enables us to change the future as well as the past chain of events. However, different examination of the system dynamics is proposed where change in the key parameters is performed while observing the change in complete future and past chain rather than observing the classical time response of the system.
The following procedure proposition emerges which enabled the anticipative formulation of the classical dynamic system: since the hyperincursive systems are hard to determine [7, 6], the developed anticipatory mechanisms should be applied, therefore the model should be a) transformed in the separated form b) provide past-future chain property and c) apply the hyperincursive structure to the studied model.

The developed model shows that by the statement of general rule of the system, the synchronization of entire feedback-anticipative chain could be gained by setting the appropriate strategy in the form of parameters value set which should be time dependant. The idea for the simulation proposed in the paper is quite different from the common paradigm. The structure of the model should yield the entire feedback-anticipative chain and the observation of the entire system response should be made. This provides new and quite challenging responses which should initiate further interest and examination of proposed model.
One of the interesting responses from the model is synchronization of the model at the certain time steps. Entire feedback-anticipative chain i.e. all point set, is synchronized according to the period of the system. The solution of the periodicity conditions for the *2-d* discrete linear cobweb map provided the means to determine the periodicity conditions. Analytical approach with *z-transformation* provides the proper way to determine the periodic solutions. The emergence of Farey tree as the rational fraction representation yields the organization of the periodicity solutions. The developed model shows that by the statement of general rule of the system, the synchronization of entire feedback-anticipative chain could be gained by setting the appropriate strategy in the form of parameters value set which should be time dependant. The bifurcation experiment with the nonlinear mapping provided the example of periodicity transposition to the systems of higher complexities. Period 6 has been determined as one of the most stable periodic solution, which has been explicitly shown by the analysis of system's attractors.

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