High Strong Order Implicit Runge-Kutta Methods for Stochastic Ordinary Differential Equations

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Abstract
The modelling of many real life phenomena for which either the parameter estimation is difficult, or which are subject to random noisy perturbation, is often carried out by using stochastic ordinary differential equations(SODEs). In this paper, a class of high strong order implicit Runge-Kutta methods for SODEs is introduced.

Keywords: stochastic differential equations; rooted trees theory; Runge-Kutta methods for $ODEs$ and $SODEs$
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1. Introduction

Consider the autonomous ordinary differential equation (ODE)

$$y'(t) = f(y(t)), \quad y(t_0) = y_0, \quad y \in \mathbb{R}^m. \tag{1}$$

The autonomous $Itô$ stochastic version of (1) can be written in differential form as

$$dy = f(y)dt + g(y)dW, \quad y(t_0) = y_0, \quad y \in \mathbb{R}^m. \tag{2}$$

Here f is an m-vector-valued function, g is an $m \times p$ matrix - valued function and $W(t)$ is a p-dimensional process having independent scalar Wiener process components $(t \geq 0)$, and the solution $y(t)$ is an m-vector process. The integral formulation of (2) can be written as

$$y(t) = y_0 + \int_{t_0}^{t} f(y(s))ds + \int_{t_0}^{t} g(y(s))dW(s) \ , \tag{3}$$

where the second integral in (3) is an $Itô$ stochastic integral (see[8,9]) with respect to the Wiener process $W(t)$. If the autonomous version of an $Itô$ stochastic ordinary differential equation (SODE) given by (2) then the related Stratonovich SODE is given by

$$dy = \bar{f}(y)dt + g(y)odW, \quad y(t_0) = y_0, \quad y \in \mathbb{R}^m, \tag{4}$$

where

$$\bar{f}(y) = f(y) - \frac{1}{2}g'(y)g(y).$$

In other words two differential equations (2) and (4) , under different rules of calculus, have the same solution. There are many different methods to solve these kinds of differential equations (see, for example [9,10,11,12]).

An outline of this paper is as follows: In section 2 a discussion on Runge-Kutta methods, especially implicit and semi-implicit Runge-Kutta methods for $SODE_s$ based on rooted trees theory is introduced(see [6,7]). In section 3 a new class of semi-implicit Runge-Kutta methods for $SODE_s$ is constructed. Numerical results are reported in section 4.

2. Runge-Kutta Methods for SODE$_s$

A s-stage Runge-Kutta method for calculating a numerical approximation to the solution of an autonomous ODE (1) is given by the recursive formula

$$\begin{aligned} Y_i &= y_n + h\sum_{j=1}^{s} a_{ij} f(Y_j) \quad i = 1, 2, \ldots, s \\ y_{n+1} &= y_n + h\sum_{j=1}^{s} b_j f(Y_j) \end{aligned} \tag{5}$$

which can be represented in tableau form:

$$\begin{array}{c|cccc} & a_{11} & a_{12} & \ldots & a_{1s} \\ & a_{21} & a_{22} & \ldots & a_{2s} \\ & \vdots & & & \\ & a_{s1} & a_{s2} & \ldots & a_{ss} \\ \hline & b_1 & b_2 & \ldots & b_s \end{array} \tag{6}$$

In tableau (6) if we do not require that the numbers a_{ij} for all i, j with $j \geq i$, are zero, then the assaciated methods of this general type will be called implicit Runge-Kutte methods, however if $a_{ij} = 0$ for $j \geq i$ the corresponding method known as an explicit Runge-Kutta method and if $a_{ij} = 0$ for $j > i$, the corresponding method is known as a semi-implicit or semi-explicit Runge-Kutta method(see[7]).

For an autonomous Stratonovich $SODE$ (4) we obtain by a straight forward generalization of (5) the class of methods

$$\begin{aligned} Y_i &= y_n + h\sum_{j=1}^{s} a_{ij} f(Y_j) + J_1\sum_{j=1}^{s} b_{ij} g(Y_j) \quad i = 1, 2, \ldots, s \\ y_{n+1} &= y_n + h\sum_{j=1}^{s} \alpha_j f(Y_j) + J_1\sum_{j=1}^{s} \gamma_j g(Y_j), \end{aligned} \tag{7}$$

where $J_1 = \int_{t_n}^{t_n+1} odW$ is the increment of the Wiener process from t_n to t_{n+1}. which can be represented in the tableau form:

$$\begin{array}{c|cccc|cccc} & a_{11} & a_{12} & \dots & a_{1s} & b_{11} & b_{12} & \dots & b_{1s} \\ & a_{21} & a_{22} & \dots & a_{2s} & b_{21} & b_{22} & \dots & b_{2s} \\ & \vdots & & & \vdots & \vdots & & & \vdots \\ & a_{s1} & a_{s2} & \dots & a_{ss} & b_{s1} & b_{s2} & \dots & b_{ss} \\ \hline & \alpha_1 & \alpha_2 & \dots & \alpha_s & \gamma_1 & \gamma_2 & \dots & \gamma_s \end{array} \tag{8}$$

Theorem 1:A stochastic Runge-Kutta method of the form (7) has maximum strong order 1.5, for any number of stages s. The methods with optimal principal error coefficients is of strong order 1.5,if:

$\alpha^T(e, b) = (1, \frac{1}{2})$,

$\gamma^T(e, c, b, b^2, Bb) = (1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{6})$.

Here $e^T = (1, \dots, 1)$, $c = Ae$, $b = Be$.

Proof: see[4].

To break this order barrier, the class of methods (7) has to be modified in some way so as to include further multiple stochastic integrals (see[4]) of the stochastic Taylor formula apart from just J_1. This has been done by K. Burrage and P.M. Burrage (see[2]).They proposed the following class of methods:

$$\begin{aligned} Y_i &= y_n + h\sum_{j=1}^{s} a_{ij} f(Y_j) + \sum_{j=1}^{s}(b_{ij}^{(1)} J_1 + b_{ij}^{(2)} \frac{J_{10}}{h}) g(Y_j) \quad i = 1, 2, \dots, s \\ y_{n+1} &= y_n + h\sum_{j=1}^{s} \alpha_j f(Y_j) + \sum_{j=1}^{s}(\gamma_j^{(1)} J_1 + \gamma_j^{(2)} \frac{J_{10}}{h}) g(Y_j), \end{aligned} \tag{9}$$

where $J_1 = \int_{t_n}^{t_n+1} odW$ and $J_{10} = \int_{t_n}^{t_n+1}\int_{t_n}^{s_2} odW_{s_1} ds_2$. which can be represented in tableau form:

$$\begin{array}{c|cccc|cccc|cccc} & a_{11} & a_{12} & \dots & a_{1s} & b_{11}^{(1)} & b_{12}^{(1)} & \dots & b_{1s}^{(1)} & b_{11}^{(2)} & b_{12}^{(2)} & \dots & b_{1s}^{(2)} \\ & a_{21} & a_{22} & \dots & a_{2s} & b_{21}^{(1)} & b_{22}^{(1)} & \dots & b_{2s}^{(1)} & b_{21}^{(2)} & b_{22}^{(2)} & \dots & b_{2s}^{(2)} \\ & \vdots & & & \vdots & \vdots & & & \vdots & \vdots & & & \vdots \\ & a_{s1} & a_{s2} & \dots & a_{ss} & b_{s1}^{(1)} & b_{s2}^{(1)} & \dots & b_{ss}^{(1)} & b_{s1}^{(2)} & b_{s2}^{(2)} & \dots & b_{ss}^{(2)} \\ \hline & \alpha_1 & \alpha_2 & \dots & \alpha_s & \gamma_1^{(1)} & \gamma_2^{(1)} & \dots & \gamma_s^{(1)} & \gamma_1^{(2)} & \gamma_2^{(2)} & \dots & \gamma_s^{(2)} \end{array} \tag{10}$$

The rest of this section is concerned with the problem of determining the strong order of convergence of stochastic Runge-Kutta methods (9). In the case of Runge-Kutta methods for deterministic problems the order of accuracy is found by comparing the Taylor series expansion of the approximate solution to the Taylor series expansion of the exact solution over one step assuming exact initial values.In 1963 Butcher introduced the theory of rooted trees in order to compare these two Taylor series expansion in a systematic way (see[7]).K. Burrage and P.M. Burrage have extended this idea of using rooted trees to the stochastic setting. They used the set of bi-coloured rooted trees, i.e., the set of rooted trees with • (τ for deterministic) and o (σ for stochastic) nodes to derive a Stratonovich Taylor series expansion of the exact solution and a Stratonovich Taylor series expansion of the approximation defined by the numerical method (9). By comparing these two expantion, they could prove the following theorem:

Theorem 2: The stochastic Runge-Kutta method (9) is of stronge order 2,if:

$\alpha^T(d,b) = (1,0)$,

$\gamma^{(1)T}(c, b^2, B^{(1)}b, d^2, B^{(2)}d) = (1, \frac{1}{3}, \frac{1}{6}, -2\gamma^{(2)T}bd, -\gamma^{(2)T}(B^{(2)}b + B^{(1)}d))$,

$\gamma^{(2)T}(c, b^2, B^{(1)}b, d^2, B^{(2)}d) = (-1, -2\gamma^{(1)T}bd, -\gamma^{(1)T}(B^{(2)}b + B^{(1)}d), 0, 0)$.

Here $e^T = (1, \ldots, 1)$, $c = Ae$, $b = B^{(1)}e$,$d = B^{(2)}e$.

Proof: see[4].

3. Implicit and Semi-Implicit Runge-Kutta Methods for SODE$_s$

In 2000 the author and Prof.M. Mohseni generalized the explicit methods satisfying (7) were derivation by K.Burrage and P.M. Burrage (see[2]) to semi-implicit and implicit methods (see[1]). More precisely we used theorem 1 and introduced the semi-implicit and implicit methods of strong order 1.5 with minimum principal error.Semi-implicit 2-stage stochastic Runge-Kutta methods are shown in tableaux (11-a) and (11-b), and were referred to "SIM" class:

$$\begin{array}{c|cc|cc}
 & (3+\sqrt{3})/6 & 0 & (3+\sqrt{3})/6 & 0 \\
 & -\sqrt{3}/3 & (3+\sqrt{3})/6 & -\sqrt{3}/3 & (3+\sqrt{3})/6 \\
\hline
 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{array} \quad (a) \qquad (11)$$

or

$$\begin{array}{c|cc|cc}
 & (3+\sqrt{3})/6 & 0 & (3-\sqrt{3})6 & 0 \\
 & -\sqrt{3}/3 & (3+\sqrt{3})/6 & \sqrt{3}/3 & (3-\sqrt{3})/6 \\
\hline
 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{array} \quad (b)$$

Implicit 2-stage stochastic Runge-Kutta methods are shown in tableaux (12-a) and (12-b), and were referred to "IM" class:

$$\begin{array}{c|cc|cc}
 & \frac{1}{4} & (3-2\sqrt{3})/12 & \frac{1}{4} & (3-2\sqrt{3})/12 \\
 & (3+2\sqrt{3})/12 & \frac{1}{4} & (3+2\sqrt{3})/12 & \frac{1}{4} \\
\hline
 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{array} \quad (a) \qquad (12)$$

or

$$\begin{array}{c|cc|cc}
 & \frac{1}{4} & (3-2\sqrt{3})/12 & \frac{1}{4} & (3+2\sqrt{3})/12 \\
 & (3+2\sqrt{3})/12 & \frac{1}{4} & (3-2\sqrt{3})/12 & \frac{1}{4} \\
\hline
 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{array} \quad (b)$$

In this section a semi-implicit stochastic Runge-Kutta method with strong order 2 will be constructed based on (9) and theorem 2 with s=3. Of course it is now necessary to construct a family of methods satisfying in theorem 2. Some simple analysis shows that this is not possible with s=2. In the semi-implicit case with s=3 and throrem 2 we has 27 free parameters and there are 18 equations to be solved. This system is solved using MAPLE and hence we conclude the following semi-implicit stochastic Runge-Kutta method with strong order 2 which are shown in tableau (13), and are referred to "SIM3" class:

$$\begin{array}{c|ccc|ccc|ccc}
 & \frac{1}{2} & & & 0 & & & 1 & & \\
 & 0 & \frac{1}{2} & & -\frac{4}{3} & 0 & & \frac{5}{3} & -\frac{2}{3} & \\
 & \frac{55}{18} & 0 & 1 & 0 & \frac{4}{9} & \frac{8}{9} & 0 & \frac{5}{6} & \frac{1}{6} \\
\hline
 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{3}{16} & -\frac{3}{32} & \frac{9}{32} & \frac{9}{16} & -\frac{9}{32} & -\frac{9}{32}
\end{array} \qquad (13)$$

Certainly, it is possible to satisfy theorem 2 in the implicit case with s=3 or in the semi-implicit and implicit case with s=4. But the large number of free parameters makes solving the similar systems difficult.

4. Numerical Results

In this section,numerical results from the implementation of 5 methods are presented. These methods are "PL", "R2", "SIM", "IM" and "SIM3". The first 4 methods taken from [1] and hence if $g_1 \sim N(0,1)$ and $g_2 \sim N(0,1)$, then for stepsize h, $J_1 = \sqrt{h}g_1$ and $J_{10}/h = \frac{\sqrt{h}}{2}(g_1 + \frac{g_2}{\sqrt{3}})$. The above methods will be implemented with constant stepsize on two problems taken from [9], for which the exact solution terms of a Wiener process is known. In order to improve the results of employing the "SIM", "IM" and "SIM3" methods at each step, we use an iteration scheme [1] with starting values come from the "PL" or "R2" methods.

For both problems and all methods, 500 trajectories are computed at each stepsize. The implementation determines the average error for each stepsize at the end of the interval of integration for each method.

Test problem 1. ([8, equation 4.4.31])

$$dy = -a^2 y(1-y^2)dt + a(1-y^2)dW, \quad y(0) = y_0, \quad t \in [0,1],$$

with exact solution

$$y(t) = \tanh(aW(t) + \operatorname{arctan}h(y_0))\cdot$$

In Stratonovich form, the above $SODE$ becomes

$$dy = a(1-y^2)\text{odw}\cdot$$

Table 1: global errors for test problem1, $a = 1, \epsilon = 0 \cdot 001, N = 500$

h	PL	$R2$	SIM	IM	$SIM3$
1/25	$0 \cdot 034189$	$0 \cdot 021000$	$0 \cdot 001221$	$0 \cdot 000775$	$0 \cdot 000505$
1/50	$0 \cdot 017179$	$0 \cdot 009935$	$0 \cdot 000580$	$0 \cdot 000324$	$0 \cdot 000114$
1/100	$0 \cdot 008061$	$0 \cdot 004711$	$0 \cdot 000297$	$0 \cdot 000091$	$0 \cdot 000068$
1/200	$0 \cdot 003850$	$0 \cdot 002343$	$0 \cdot 000188$	$0 \cdot 000038$	$0 \cdot 000014$

Table 2: global errors for test problem1, $a = 0 \cdot 5, \epsilon = 0 \cdot 001, N = 500$

h	PL	$R2$	SIM	IM	$SIM3$
1/25	$0 \cdot 003607$	$0 \cdot 001469$	$0 \cdot 000058$	$0 \cdot 000021$	$0 \cdot 000013$
1/50	$0 \cdot 001808$	$0 \cdot 000712$	$0 \cdot 000032$	$0 \cdot 000015$	$0 \cdot 000010$
1/100	$0 \cdot 000861$	$0 \cdot 000330$	$0 \cdot 000019$	$0 \cdot 000011$	$0 \cdot 000007$
1/200	$0 \cdot 000428$	$0 \cdot 000156$	$0 \cdot 000010$	$0 \cdot 000008$	$0 \cdot 000002$

Test problem 2. ([8, equation 4.4.46])

$$dy = -(\alpha + \beta^2 y)(1 - y^2)dt + \beta(1 - y^2)dW, \quad y(0) = y_0, \quad t \in [0, 1],$$

with exact solution

$$y(t) = \frac{(1 + y_0)\exp(-2\alpha t + 2\beta W(t)) + y_0 - 1}{(1 + y_0)\exp(-2\alpha t + 2\beta W(t)) + 1 - y_0}.$$

In Stratonovich form, the above $SODE$ has the form

$$dy = -\alpha(1 - y^2)dt + \beta(1 - y^2) o dW\cdot$$

Table 3: global errors for test problem2, $\alpha = 1 \cdot 0, \beta = 0 \cdot 01, \epsilon = 0 \cdot 001, N = 500$

h	PL	$R2$	SIM	IM	$SIM3$
1/25	$0 \cdot 007381$	$0 \cdot 000111$	$0 \cdot 000007$	$0 \cdot 000003$	$0 \cdot 000000$
1/50	$0 \cdot 003666$	$0 \cdot 000027$	$0 \cdot 000001$	$0 \cdot 000000$	$0 \cdot 000000$
1/100	$0 \cdot 001827$	$0 \cdot 000007$	$0 \cdot 000000$	$0 \cdot 000000$	$0 \cdot 000000$
1/200	$0 \cdot 000912$	$0 \cdot 000001$	$0 \cdot 000000$	$0 \cdot 000000$	$0 \cdot 000000$

Table 4: global errors for test problem2, $\alpha = 1 \cdot 0, \beta = 2 \cdot 0, \epsilon = 0 \cdot 001, N = 500$

h	PL	$R2$	SIM	IM	$SIM3$
1/50	$0 \cdot 179303$	$0 \cdot 143407$	$0 \cdot 039636$	$0 \cdot 029369$	$0 \cdot 017451$
1/100	$0 \cdot 083476$	$0 \cdot 064094$	$0 \cdot 013703$	$0 \cdot 012442$	$0 \cdot 009212$
1/200	$0 \cdot 051587$	$0 \cdot 039694$	$0 \cdot 009300$	$0 \cdot 007101$	$0 \cdot 005013$
1/400	$0 \cdot 022484$	$0 \cdot 018316$	$0 \cdot 003835$	$0 \cdot 001939$	$0 \cdot 000728$

5. Conclusions

In this paper, we have constructed an implicit Runge-Kutta method of strong order 2.

Our future work should be based on the construction of implicit Runge-Kutta methods for $SODE_s$ with two or more Wiener processes.

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