THE STUDY OF DYNAMICAL SYSTEMS' NONLINEAR CHARACTERISTICS

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ABSTRACT

Nonlinearity is the source of complexity. It gives rise to the change of system behaviors, the evolution of structures and such phenomena as bifurcation,catastrophe,and even chaos. It is these phenomena,dovetailed with others , that weave out our multicolour and multifold world synergetically. With the development of science and technology, people become more and more interested in and capable of the study of nonlinearity so as to shed light on the nature of the world. In order to deal with nonlinearity more systematically, this paper elaborates a comprehensive description for the dynamical system. Then, we focus on the relationships between the characters of nonlinearity . We have successfully expounded some controversial concepts, cast new light on some important relations, and unified several concepts which are the central topics of many modern theories.

SYSTEM DESCRITION

As we all know, system dynamics defines a system as a set in which its interacting and interdiffering parts organically link together so as to perform a certain goal. To lay a fundation for further development, we frame an axiomatical definition for dynamical system. Suppose V is a set which concludes the necessary elements of a system. X is the state space of the system, which consists of the description of everything one needs to know in order to describe how the system will change.From the viewpoint of S.D., the summer of the elements of the system will be greater than the whole because of the existance of interaction and interlimitation, namely,structure, between the elements.So, let R represents the set of all feedback relations among the partsand sub systems.Then, (X,R) can statically describe the system.

To represent the system dynamically, we introduce state transition function. Let X denote the state space,a state transition function,T, is a function from X * V to X.T(x,t)gives us the state of the system at time t if the system was in state x at time 0.

Thus, we reach:

Def. 1. A dynamical system is the trielement of

(X ,R, T) ....(1)

To facilitate study, as X,R,T have their own components respectively, we have:

Def. 2. A dynamical system can be described as

(Vi,X,R,Xo,F,I,P,G, t)...........(2)

where Vi is the set of subsystems of the system, X is the state variable, R is the set of all feedback relations, Xo is the initial value of X, F is the exterior and interior force, I denotes information, P represents parameter,including decision parameter,G is differential operators (indicating space gradient) , and t is time.

In this way, not only can we describe the complex nonlinear system, but we also can unify many important concepts underit and distinguish them clearly. For example, we are able to coalesce all the concepts of stability which have appeared or may emerge. Traditionally, we have

Def.3. Liaponov Stability
Let $dx/dt = f(x)$ be a dynamical system on X with equilibrium x^*. If $x(t)$ approaches x^* as t goes to infinity,for any initial conditions,then we call the system is stable at x^*.

A newer concept of stability is:

Def. 4. Structure Stability

Let $f : X \text{---}> R^n$ define a vector field on some state space X. We say this system is structurally stable if small perturbations in f do not change the topological structure of the vector field.

It is often of considerable interest ,especially in the study of chaos,to know how solution curves behave as we vary the initial conditions.

Def. 5. Initial Value Stability
Let $f : X \text{----}> R^n$ and let $dx/dt = f(x)$ define a system of differential equtions with initial conditions $x(0)$. If there is a $d>0$ such that ,when $x(0)$ and $y(0)$ are small enough, the absolute value of $x(t,x(0))$ and $y(t,y(0))$ is less thand,we call the system is stable to initial value.

These three definitions of stability are what we know so far.They are of great significance in the study of nonlinearity. Now, we put them under one definition to see the natures both in common and in difference.

Def. 6. Stability is the nature of a system to approach its original state after disturbtion.

According to this definition, if we choose time t in (2) to change a little,we have Def.3., i.e., Liaponov stability; if

,instead ,we select P in (2), we reach Def. 4.; and so is Def. 5 for x(0).

It is obvious,from the above analysis,that the three definitions are different in nature. From (2), we can also get different definitions of stability or other in terms of need.

NONLINEARITY AND BIFURCATION

The world we live in, both the natural world and the social world, is very complicated.It is of various behaviors and multicolour. It is the organic dovetail of deterministics and stochastics. One of the most essential charaters of this complexity is its evolution: it can perform shiftness from one phase to another quite different phase.

Example 1 Bernard Flow[1]

This is a well known model.

When dT = 0 ,<======>equilibrium (disorder)
Increase dT, for small fluctuation=====>stable
When dT overpasses the valve dT(c(0)), ====> emerges order
If increase dT even more ====> chaos

This complexity,to a great extent, results from the system's nonlinearity. Nonlinearity is the source of versitility and innovation. See Figure 1. From the figure, we can see, because of nonlinearity, there must be some b* such that f(b*) has two or more values: f1=f(b1*), f2 = f(b1*). This ,in turn, results in choosing : f1 or f2?

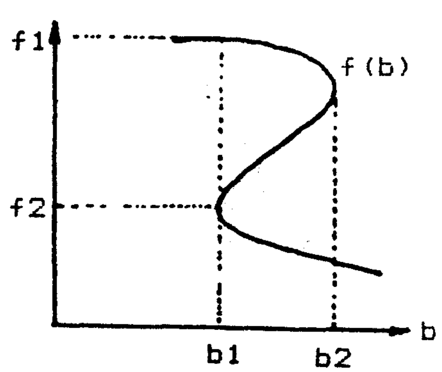

Figure 1

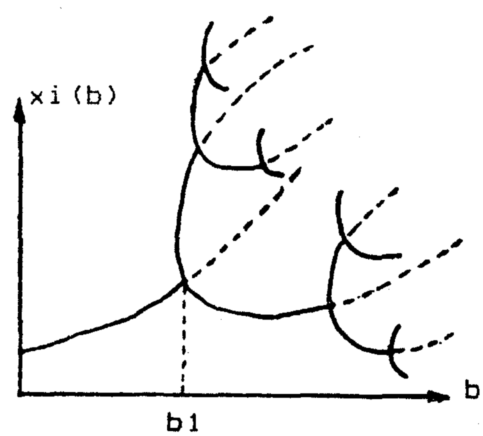

Figure 2

The choosing process is definitely determined by chance. For instance, in the example of Bernard Flow, it is totally random as to appear which order : right rotation or left rotation.What is more, in nonlinear system, a very small factor can cause the value to change dramatically. As in Fig.1, at O(bi,d),d>0,for a very small db, df jumps suddenly :

(f1-f2)/(bi+db-bi) ----> infinite.

This causes the system to change catastrophely at b(i). From Fig.1, we can also visualize that the indispensible link between nonlinearity and complexity is bifurcation,i.e., the relation is as following :

Nonlinearity ===> Bifurcation ===> Complexity

Def. 7 For equation f(x,b) = 0 ....(3), b belongs to R. If there is a point (0,b(0)) satisfies : for any d>0,there is a uncommon solution (x b),x≠0, in O((0,b(0)),d),then we call (0,b(0)) a bifurcation point.

Its meaning is conspicuous: from f(x,b) = 0,we can derive xi(b) such that f(xi(b),b) =0, Due to nonlinearity, when b has a very small altertation at some certain point, say, b(1), there will be great change:xi(b),which is single,======>xi+(b),which has two values. xi(b) are the seeking goals of the system. Tracing the bifurcation chart (Fig.2),we know where the system will go.

Now, we give out some useful results:

Theorem 1. If (0,b(0)) is a bifurcation point, then ,df/dx(0,b(0)) = 0

<proof>. If df/dx(0,b(0))≠0,from the existance theorem of implying function,it can be inversed.So, there is a d(0)>0,in O((0,b(0)),d(0)),equation (3)'s solution X(b)=0 <==> (0 ,b(0)) is not the bifurcation point. Theorem 1 is important. It gives out an easy way to judge bifurcation. Unfortunatlly, it is not sufficient.

For eamaple.
```
                    | (1-b)x-y^3
          f(x,y,b)= |.(x^3+(1-b)y

then      F(0,0;1)=0
          F'(0,0;1)=0
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But (0,0;1) is not the bifurcation point.
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   since      F(x,y;1)=0
          (1-b)x-y^3=0
            <==> x^3+(1-b)y=0 <==> x=y=0
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Theorem 2. Linearity ====> no bifurcation

<proof>. For system dx/dt = f(x), if f(x) is linear, f(x) = ax+b (a≠0, b≠0)

so, dx/dt = ax+b ====> x=(EXP(at))*(INT.(bEXP(-at)+c))
= b + c*EXP(at)

This is exponential mode, no bifurcation. (if bifurcation, from theorem 1, dx/dt for some value of the parameter, but, dx/dt=c*aexp(at)≠0, so no bifurcation)

Theorem 3. Hopf Theorem (see [1])

It gives out the situation from a focus to a limit cycle.

AN INSIGHT ON THE WAYS TO COMPLEXITY

As we stated before, nonlinearity makes it possible for system (1) to bifurcate,which results in complexitysince bifurcation can give rise to the loss of structure stability,the shift of dorminant loop, catastrophe, and chaos.So far, there are many discussions on each of these phenomena.But,few attention has been paid to the relationships between them.In fact, there are strong linkages between them.By means of bifurcation, it is possible for us to cast light on these linkages so that we can have a better understanding of the charateristics of nonlinearity such as aperioical solution, bifurcation, chaos, stability and attractor.

1. Bifurcation versus Structure Stability

One of the most important natures of bifurcation is that, the behavior of the system will show qualitative alteration when the parameter (P in (2)) changes a little. First of all, from section I.,structure stability is definitely different from Liaponov Stability.

Example 2

d(dx/dt)/dt + x = 0
set x1=x, x2=dx/dt,
x=(x2,-x1)

If given a small enough disturbence dx=(0 ,-ex2), e<<1, then x+dx :d(dx/dt)/dt+e(dx/dt)+x=0 for any e>0, the behaviours are quite different. So,not structure stable. But for system d(dx/dt)/dt+adx/dt+x =0 a≠0, structure stable a=0, not so, a = 0 is the bifurcation point. (see figure 3) Thus, we can study structure stability by bifurcation.

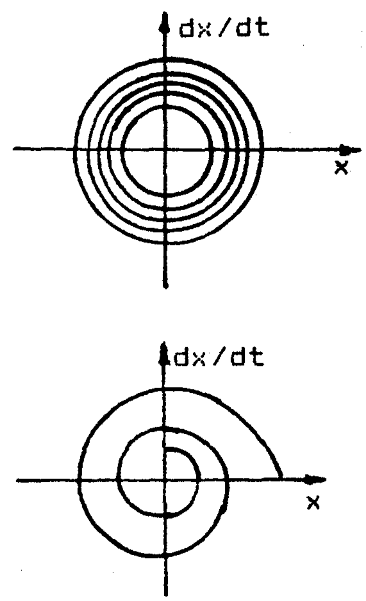

Figure 3

Theorem 4. (Y.Z.P.)

Structure Stability <===> No Bifurcation

<proof>. In order to prove this theorem, we first have a deeper discussion of the concept of structure stability.
Let us turn to a vector field dX/dt=f(x). We will take the state space of this system to be H^n . Let U be the space of all continnously differentiable functions fron H^n to R^n , and we endow U with the standand C' norm, i.e., two functions are close if their values are close and their derivatives are close. We can then think of a perturbation off as being a choice of any function in some E-ball around f.

We want the topological structure of X=f(x) to be invariant with respect to small perturbation of f. So, we introduce the relevant concept, that is, topological equivalence. Roughly speaking, the flow of two dynamical systems on H^n are topologically equivalent if there is a homeomorphism h: H^n-->H^n that carries the orbits of one flow into the orbits of the other. We can think of this homeomorphism as being some continuous change of co-ordinates, so that topological equivalence of two flows just means that we can find a continuous change of coordinates so that one flow looks like the other.

Thus, we can describe sturcture stability as following:

A dynamical system dX/dt=f(x) on H^n is structurally stable if there is some neighborhood of f such that for every functinn g in that neighborhood, the flow induced by dX/dt=g(x) is topologically equivalent to the flow of f. Loosely speaking, a dynamical system is structurally stable if small perturbations in the underlying function f do not change the qualitative nature of the flow.

If a system is structure stable everywhere but there is a point (o,b(o)) which is the bifurcatin point of the system. Then, in the neighborhood of (o,b(o)), there must be another orbit of the system which is qualitatively different from the above one. That is, there is not any homeomorphism ho which is able to carry the first orbit to the second. So, the system is not structure stable at (o,(o)). This is contradict to the assumption. So, we reach:

Structure stability ==> No Bifurcation.

On the other hand, if the system have not any bifurcation points, but the system experiences the process of loss of structure stability at a certain point. We can easily trace this point to find that this is a bifurcation point. Of course, this will not be the case. That is to say,

No Bifurcation ==> Structure Stability

Structure stability is of great significance both in society and nature.The existance of structure stability is the prerequisite for system to exist. The loss of structure stability is the condition for system to change. It causes the dominant loops of the system to shift so the system can evolute. At the same time,the loss of structure stability may lead to chaos. It also expounds why system dynamics models are often insensitive.

2. Bifurcation versus Shift of Dominant Loop

In the study of dynamical system,it is of great importance to study system's dominant loops at a given time, so that we can get the developing picture of the system in our hand. Here , we shall study SDL through bifurcation and structure stability. (Traditionally, we study SDL by eigenvalue and Lioponov methods, but they are abstract and, to some extent, not tangible)

Def.8. for system dx/dt=f(x), f belongs to C(V), we define Loop Polarity=sign((dx/dt)/dx)

Theorem 5. SDL<=>change sign((dx/dt)/dx)
<=>change sign(df/dx)
<=>there is xi* such that (df/dx)(xi*)=0
<=>xi* are f(x)'s max or min points.

The proof is easy.

Meaning of theorem 5: From this theorem, we can study the shift of dominant loops by a new tool. In 1984 SD International Conference, a paper put forward the guess that

Bifurcation <===> SDL (see[3]).

By our study, this guess is not correct.The rational relation between them is :

Theorem 6. Bifurcation ====> SDL
<proof>. from theorem 2,
Bifurcation=====> df/dx(0,b(0)) =0
from theorem 5, df/dx = 0 <===> SDL
So, bifurcation ====>df/dx = 0 <===> SDL
But,this is not sufficient,i.e., we can not have df/dx = 0 ===> Bifurcation. (see example 2)

From theorem 4 and theorem 6, we have :

Theorem 7. The loss of structure stability =====> SDL
<proof>. Loss of Structure Stability <===> Bifurcation ===> SDL

3. Bifurcation versus Catastrophe Theory

In order to study how structure change occur, we need catastrophe theory. Let's consider some dynamical system given by f:X*(A)--> R^n , dx/dt=f(x,b). Here the system is thought of as parametrized by some parameters a=(a1,...,ar). Now, suppose we think of the parameters as changing slowly over time. Most of the time, there will not be radical changes in the qualitative nature of the dynamical system. Howerer, sometime we do get real structural change.

For example, consider:

Example 3. dx/dt=x^2+a

If a>0, there are no equilibria of this system.
If a=0, there is only one equilibrium : x*=0
If a<0, two equilibria : x1*=-SQRT(a)
x2*=SQRT(a)

The topological nature of this system undergoes a radical change as a passes through zero. We say b=0 is a catastrophe point for the system (dx/dt)=x^2+a

Theorem 8. Bifurcation point <==> Catastrophe point.

Theorem 9.Loss of Structure Stability <==> Catastrophe Point

4. Bifurcation versus Chaos

Bifurcation means, in mathematics, that a very small change in initial values will cause a fundamental qualitative difference in the long run. (see Fig.1) This is, roughly speaking, chaos. Chaos is a newly developed studying field. It has great interrelation with almost all the scientific branches which study nonlinearity. By our study, we find that bifurcation not only redounds to the resolution of chaos, but it is also the necessary condition for chaos to occur.

Ending Words

Nonlinearity study is an arduous task.We can imagine how difficult it is when we consider that only a very small portion of the majestic mathematics edifice deals with non linearity, and that nearly all models in socio-economic sciences directly or indirectly handle problems with linear theory. Fortunately, more and more calibers have become engrossed in this field of late and we can envesion a bright future soon.

REFERENCES

[1] H. Haken: Synergetics 1988.4 , Shanghai Science Press.

[2] B. Hassard and Y.H.Wan: Bifurcation Formula Theory Journal of Mathe. Analysis and Applications 63, 297-312 (1978)

[3] Proceedings of the 1984 International Conference of S.D. Society.