To Main Proceedings Document

Implementing formal model analysis

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Abstract:

There have been a number of important contributions to the field of system dynamics in the recent past. One of the most exciting has been the development of eigenvalue elasticity theory (Forrester 1982, Kampmann 1996), which promises to increase our capacity to understand our models, enhancing our ability to get insight from modeling. In this paper, we critically review the current state of knowledge and provide an improved approach to determining loop gain elasticities.

1. Introduction

There have been a number of important contributions to the field of system dynamics in the recent past. A significant one was the introduction of advanced software environments that made use of modern graphical computers. IThink/STELLA, Vensim, and Powersim link diagrams to equations, simplifying considerably the process of building models. These software environments have more recently permitted new ways of running models by making it possible host models on the world wide web and by making it possible to create interactive, multimedia management games. The user friendly software environments have made modeling more accessible, and the number of new practitioners in system dynamics has consequently multiplied.

Although the improved software has lowered the barriers to entry to the field and reduced the cost of creating and running models, analysis of models remains difficult for the inexperienced and time-consuming for everyone. The creators of Vensim made a valuable contribution by automating and illuminating the process of causal tracing. But, beyond this there has been essentially no advance in analytical tools since 1960, when non-overlapping data plots were introduced in DYNAMO.[1] The analysis of models in a rigorous, quick, reliable, and standard manner is still today one of the most important and challenging problems in system dynamics.

A research area that promises to increase our capacity to analyze our models is the idea of eigenvalue elasticity.[2] (N. Forrester 1982, Kampmann 1996) The process essentially is to linearize the model at a point in time, calculate the eigenvalues and then consider how the eigenvalues change as link gains change for the linearized system. The resulting understanding of how link-gains impact eigenvalues can be used directly in deciding which links to strengthen or weaken, or indirectly to figure out which loops contribute most heavily to the eigenvalues. Software capable of carrying out the process was developed some fifteen years ago in an experimental version of DYNAMO. Unfortunately, too few people used it, and eventually Pugh-Roberts discontinued the experimental version without ever determining how useful the approach would be on real-world problems.

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[1] The new software environments, especially Vensim, have also increased the ease of incorporating data into models and have provided new tools for model calibration and policy optimization.

[2] Another promising area for formal model analysis is pathway participation (Mojtahedzadeh 1996), but we have not reviewed this technique here.

Forrester showed that a complete description of *link* elasticities allows one in principle to calculate *loop* elasticities. This suggestion though never implemented in software, promised to provide an answer to how model structure, that is a set of feedback loops, determines model behavior.

The particular calculation that Forrester suggested is actually not feasible. As he realized later, Forrester's suggested approach results in a system of equations that is over-determined – an effect of the fact that the number of loops increases much faster than the number links. Kampmann discovered that a small subset of loops is sufficient to uniquely describe eigenvalues (i.e. the behavior) of a system dynamics model. (Kampmann 1996) Using such an "independent loop set" produces a smaller system of equations, a system that can be solved.

A problem remains even with Kampmann's approach. Usually, there will be a number of different independent loop sets. While any one of them can be used to discover how eigenvalues change as members of the set change, not all loops will be members of the set. The question remains which loopset to choose, or whether an arbitrarily chosen independent loopset can provide intuitively-appealing insight into a system.

In this paper we present an alternative way to compute *loop*-gain eigenvalue elasticities. This approach provides information on every loop in the model. We sidestep the problem of an over-determined set of equations by eliminating the intermediate step of using the relatively small number of link elasticities to get the relatively large number of loop gains. Instead we express the characteristic equation directly in terms of *loop* gains; and then use this equation to go directly to *loop* gain elasticities. Our approach calculates a unique gain elasticity for every loop in the system, not just for a subset of loops.

2. Eigenvalue elasticity theory

The formal structure of a system dynamics model with a vector of state variables $\mathbf{x}(t)$, where $\mathbf{x}(t) = (x_1, x_2, \ldots, x_n)'$, a vector of auxiliary variables $\mathbf{y}(t)$, where $\mathbf{y}(t) = (y_1, y_2, \ldots, y_p)'$, a vector of nonlinear functions $\mathbf{f}(t)$, where $\mathbf{f}(t) = (f_1, f_2, \ldots, f_n)'$ and $\mathbf{g}(t)$, where $\mathbf{g}(t) = (g_1, g_2, \ldots, g_p)'$, can be represented by the set of nonlinear differential equations: (Kampmann 1996)

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{y}(t), t), \\ \mathbf{y}(t) &= \mathbf{g}(\mathbf{x}(t), \mathbf{y}(t), t). \end{aligned} \tag{1}$$

When the system is not influenced by exogenous variables, **f** and **g** are not functions of time. So, we can simplify the notation.

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{y}), \\ \mathbf{y} &= \mathbf{g}(\mathbf{x}, \mathbf{y}). \end{aligned} \tag{2}$$

Given the complexity of the system of nonlinear differential equations, we gain tractability by linearizing it around an operating point $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ where, $\bar{\mathbf{y}} = \mathbf{g}(\bar{\mathbf{x}}, \bar{\mathbf{y}})$. Linearization allows every variable (v_i) to be expressed as a linear combination of other variables (v_j ,where j = 1, 2,…, i,…,n) in the model, such that:

$$\mathbf{v}_i = \sum_j \mathbf{a}_{ij}\mathbf{v}_j, \quad \text{where,} \quad \mathbf{a}_{ij} = \frac{\partial v_i}{\partial v_j}, \tag{3}$$

and, $\mathbf{a}_{ij}$ is the partial derivative of variable $\mathbf{v}_i$ with respect to variable $\mathbf{v}_j$.

We will illustrate with a version of the familiar workforce inventory model. The model captures a simple production system. The model attempts to maintain desired inventory by adjusting production via hiring and firing workers. More precisely: Inventory integrates the difference between production and shipments. Shipments are determined by demand reduced by stock-outs, should inventory fall too low. Production depends on the workforce. And the workforce is "anchored" to the level necessary to meet expected demand. The workforce is increased above this anchor if inventory is too low and conversely workforce is decreased below the anchor if inventory is too high. Expected demand is a smooth of actual demand.

A stock and flow diagram of the model is shown below. The model is composed of three state variables, four flows, three auxiliary variables, two exogenous variables, and five constants.

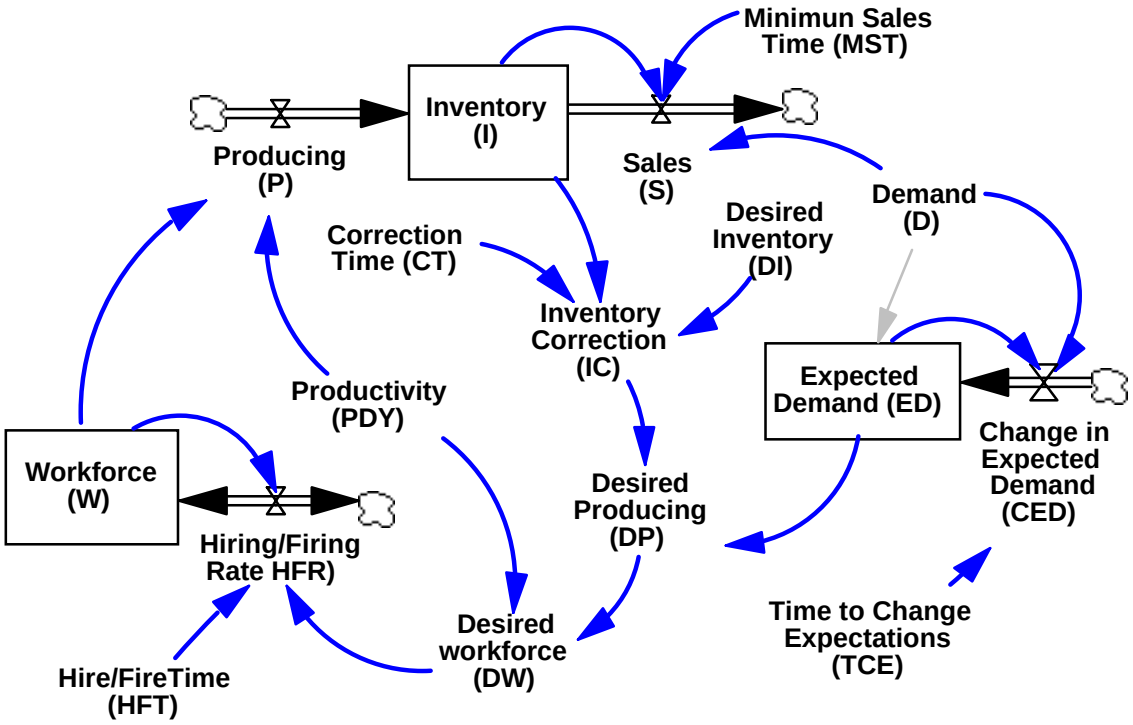


Figure 1 – Diagram of a linear system dynamics model.

$$\dot{I} = P - S = PDY \cdot W - D \qquad IC = (DI - I)/CT$$

$$\dot{W} = HFR = (DW - W)/HFT \qquad DP = IC + ED$$

$$\dot{ED} = CED = (D - ED)/TCE \qquad DW = DP/PDY$$

In terms of equation (2) $\mathbf{f(x, y)} = (f_1, f_2, f_3)'$, where, for example, $f_1(I,W,ED, IC, DP, DW) = PDY.W - D$ and $\mathbf{g(x, y)} = (g_1, g_2, g_3)'$, where $g_1(I,W, ED, IC, DP, DW) = (DI - I)/CT$.

Such a system of differential equations can be represented it in matrix form as:

$$\begin{bmatrix} \dot{\tilde{\mathbf{x}}} \\ \tilde{\mathbf{y}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{x}} \\ \tilde{\mathbf{y}} \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix}, \tag{4}$$

where the tilde represents the deviation from the operating point, the entries in matrices **A**, **B**, **C**, and **D** are given by the partial derivatives, $A_{n\,\mathbf{x}\,n} = \partial\dot{\tilde{\mathbf{x}}}/\partial\tilde{\mathbf{x}}$, $B_{n\,\mathbf{x}\,p} = \partial\dot{\tilde{\mathbf{x}}}/\partial\tilde{\mathbf{y}}$, $C_{p\,\mathbf{x}\,n} = \partial\tilde{\mathbf{y}}/\partial\tilde{\mathbf{x}}$, $D_{p\,\mathbf{x}\,p} = \partial\tilde{\mathbf{y}}/\partial\tilde{\mathbf{y}}$, and $\mathbf{b} = \mathbf{f}(\bar{\mathbf{x}}, \bar{\mathbf{y}})$.[3]

The system of differential equations in figure 1 in matrix form becomes:

$$\begin{bmatrix} \dot{I} \\ \dot{W} \\ \dot{ED} \end{bmatrix} = \begin{bmatrix} 0 & PDY & 0 \\ 0 & -1/HFT & 0 \\ 0 & 0 & -1/TCE \end{bmatrix} \begin{bmatrix} I \\ W \\ ED \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/HFT \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} IC \\ DP \\ DW \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1/TCE \end{bmatrix} D$$

$$\begin{bmatrix} IC \\ DP \\ DW \end{bmatrix} = \begin{bmatrix} -1/CT & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I \\ W \\ ED \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/PDY & 0 \end{bmatrix} \begin{bmatrix} IC \\ DP \\ DW \end{bmatrix} + \begin{bmatrix} 1/CT \\ 0 \\ 0 \end{bmatrix} DI$$

Or, simply:

$$\begin{bmatrix} \dot{I} \\ \dot{W} \\ \dot{ED} \\ IC \\ DP \\ DW \end{bmatrix} = \begin{bmatrix} 0 & PDY & 0 & 0 & 0 & 0 \\ 0 & -1/HFT & 0 & 0 & 0 & 1/HFT \\ 0 & 0 & -1/TCE & 0 & 0 & 0 \\ -1/CT & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/PDY & 0 \end{bmatrix} \begin{bmatrix} I \\ W \\ ED \\ IC \\ DP \\ DW \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 1/TCE & 0 \\ 0 & 1/CT \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} D \\ DI \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & PDY & 0 \\ 0 & -1/HFT & 0 \\ 0 & 0 & -1/TCE \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/HFT \\ 0 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} -1/CT & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/PDY & 0 \end{bmatrix}$$

[3] Note that when we have models with exogenous *constants*, we can still have functions **f** and **g** independent of time, nevertheless they can modify the differential and algebraic equations. In such cases, the exogenous constants are gathered in a new vector (**c** and **d**) and a matrix (E and F) multiplies it to capture the relationships they have with the system. Under this condition equation (4) becomes:

$$\begin{bmatrix} \dot{\tilde{\mathbf{x}}} \\ \tilde{\mathbf{y}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{x}} \\ \tilde{\mathbf{y}} \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{E} \\ \mathbf{F} \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{d} \end{bmatrix}$$

We can also represent the system in a more compact diagrammatic way using graph theoretical notation. Thus, the system in figure one becomes:

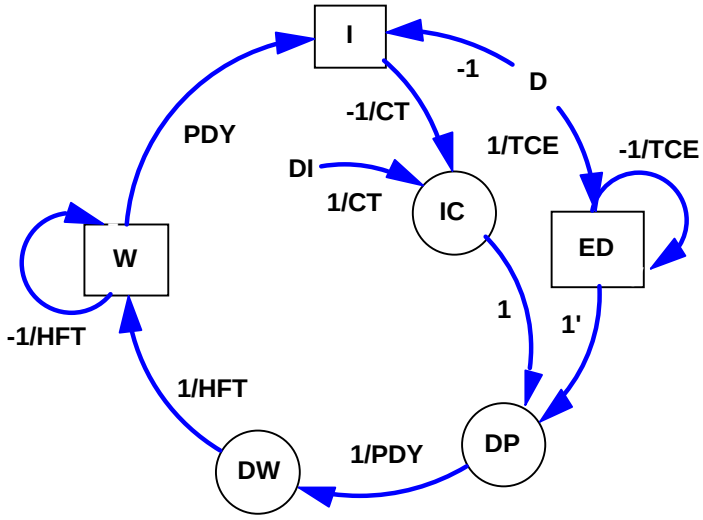


Figure 2 – A graph theoretical representation of a system dynamics model in non-reduced form.

Additionally, Any system can be represented in reduced form by eliminating $\tilde{\mathbf{y}}$ from equation (4):

$$\dot{\tilde{\mathbf{x}}} = \mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}\tilde{\mathbf{y}} + \mathbf{b},$$

$$\tilde{\mathbf{y}} = \mathbf{C}\tilde{\mathbf{x}} + \mathbf{D}\tilde{\mathbf{y}}$$

$$(\mathbf{1} - \mathbf{D})\tilde{\mathbf{y}} = \mathbf{C}\tilde{\mathbf{x}}$$

Assuming that (1-D) is invertible we can write

$$\tilde{\mathbf{y}} = (\mathbf{1} - \mathbf{D})^{-1}\mathbf{C}\tilde{\mathbf{x}}$$

Substituting into the equation for $\dot{\tilde{\mathbf{x}}}$ we have:

$$\dot{\tilde{\mathbf{x}}} = \mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}[(\mathbf{1} - \mathbf{D})^{-1}\mathbf{C}\tilde{\mathbf{x}}] + \mathbf{b}$$

And, rewriting the equation above, we obtain the result in reduced form:

$$\dot{\tilde{\mathbf{x}}} = \mathbf{J}\tilde{\mathbf{x}} + \mathbf{b}, \qquad \mathbf{J} = \mathbf{A} + \mathbf{B}(\mathbf{1} - \mathbf{D})^{-1}\mathbf{C}. \tag{5}$$

where, the matrix $\mathbf{J} = \partial\dot{\tilde{\mathbf{x}}}/\partial\mathbf{x}$ is known as the *Jacobian* of the system.

Returning to our example, the Jacobian (**J**) leads to the following relation:

$$\mathbf{J} = \begin{bmatrix} 0 & PDY & 0 \\ -1/HFT \cdot PDY \cdot CT & -1/HFT & 1/HFT \cdot PDY \\ 0 & 0 & -1/TCE \end{bmatrix},$$

The eigenvalues (λ) of the matrix **J** describe the behavior modes inherent in the model. The eigenvalues are the solutions of the characteristic polynomial **P(λ)**:

$$P(\lambda) = |\lambda I_n - \mathbf{J}| = 0 \tag{6}$$

The characteristic equation for the example provided is given by:

$$P(\lambda) = |\lambda I_3 - \mathbf{J}| = \begin{vmatrix} \lambda & -PDY & 0 \\ 1/HFT \cdot PDY \cdot CT & \lambda + 1/HFT & -1/HFT \cdot PDY \\ 0 & 0 & \lambda + 1/TCE \end{vmatrix},$$

$$= (\lambda + 1/TCE)(\lambda^2 + \lambda/HFT + 1/(HFT \cdot CT))$$

$$= \lambda^3 + (\frac{1}{TCE} + \frac{1}{HFT})\lambda^2 + (\frac{1}{TCE \cdot HFT} + \frac{1}{CT \cdot HFT})\lambda + \frac{1}{TCE \cdot HFT \cdot CT}$$

And, the eigenvalues, for the example, in terms of the link gains are:

$$\lambda_1 = -\frac{1}{TCE}$$

$$\lambda_2 = -\frac{1}{2 \cdot HFT} + \frac{1}{2}\sqrt{\frac{1}{HFT^2} - \frac{4}{CT \cdot HFT}}$$

$$\lambda_3 = -\frac{1}{2 \cdot HFT} - \frac{1}{2}\sqrt{\frac{1}{HFT^2} - \frac{4}{CT \cdot HFT}}$$

Since the Jacobian (**J**) is computed from **A**, **B**, **C**, and **D** and the entries of those matrices are the partial derivatives or the link gains (a_{ij}) in the system dynamics model, it is simple to obtain the characteristic polynomial P (λ) in terms of link gains (a_{ij}).[4] Thus, Forrester suggested measuring the sensitivity of an eigenvalue with respect to a specific link by simply computing the partial derivative of the eigenvalue with respect to the link gain. This would allow one to understand how the strength of a link could impact specific modes of behavior.

$$S_{kij} = \partial\lambda_k \Big/ \partial a_{ij} \tag{7}$$

Additionally, one could normalize the sensitivity measure to isolate the effect of the change from the sizes (values) of eigenvalues and link gains. This normalization could be obtained multiplying the sensitivity by the ratio of the link gain to the eigenvalue. He defined this measure *eigenvalue elasticity with respect to link gain* or *link gain elasticity*.

$$E_{kij} = \frac{\partial\lambda_k}{\partial a_{ij}} \frac{a_{ij}}{\lambda_k} \tag{8}$$

[4] This can be directly observed from the example above.

Although it is not common practice to write the characteristic polynomial P (λ) in terms of the loop gains (g_n) and Forrester never explicitly wrote P (λ) in terms of the loop gains, he did suggest that he could obtain the loop gain elasticities from Mason's rule. Thus, he extended the results for link sensitivity and link elasticity to loops.

$$S_{kn} = \partial\lambda_k \Big/ \partial g_n \quad \text{and} \quad E_{kn} = \frac{\partial\lambda_k}{\partial g_n}\frac{g_n}{\lambda_k} \tag{9}$$

Considering that loop gains $\mathbf{g_n}$ are determined by the product of the gains of links in the loop and that, according to Mason's rule, the characteristic equation can be written in terms of the loops gains, we obtain that the eigenvalues are a function of link gains:

$$\boldsymbol{\lambda_k = h(g_n(a_{ij}))} \tag{10}$$

Given also that any link can be a part of several distinct loops, Forrester rewrote E_{kij}:

$$E_{kij} = \sum_{over\ all\ n} \left(\frac{\partial\lambda_k}{\partial g_n}\frac{\partial g_n}{\partial a_{ij}}\right)\frac{a_{ij}}{\lambda_k} \tag{11}$$

The partial derivative of the loop gain ($\mathbf{g_n}$) with respect to the gain of a link (a_{ij}) *not* in the loop is *zero* ($\partial g_n / \partial a_{ij} = 0$). And, the partial derivative of the loop gain with respect to the gain of a link (a_{ij}) in the loop is the ratio of g_n by a_{ij} ($\partial g_n / \partial a_{ij} = g_n / a_{ij}$). Then, we can obtain a final equation for the link gain elasticity:

$$E_{kij} = \sum_{\substack{over\ all\ a_{ij} \\ in\ n}} \frac{\partial\lambda_k}{\partial g_n}\cdot\frac{g_n}{a_{ij}}\cdot\frac{a_{ij}}{\lambda_k} + \sum_{\substack{over\ all\ a_{ij} \\ not\ in\ n}} \frac{\partial\lambda_k}{\partial g_n}\cdot 0\cdot\frac{a_{ij}}{\lambda_k}$$

$$E_{kij} = \sum_{\Lambda_{ij}} \frac{\partial\lambda_k}{\partial g_n}\frac{g_n}{\lambda_k} \tag{12}$$

where Λ_{ij} is the set of all loops containing the link from v_j to v_i.

Equation (12) shows that the link gain elasticity is equal to the sum of the loop gain elasticities of all the loops composed by link a_{ij}. (N. Forrester, 1983) At this point, the theory of eigenvalue elasticity could tell how link elasticities could influence behavior, by its impact on the eigenvalues. Ultimately, it would be interesting for practitioners to be able to say how structure drives behavior. Thus, it would be a stronger result to say tell how specific loops would influence behavior. While Forrester hinted at the problem, he didn't provide a clear the solution.

In 1996, Kampmann introduced a few developments to the theory that had the potential to answer how specific loops could drive model behavior. First, Kampmann formalized the existing notation in the field through the introduction of graph theory. This

allowed him to describe the topology of a system dynamics model in a simpler way using directed graphs.

Then, he explored the interdependencies between the number of loops and links in a maximally connected system. He observed that the number of loops increased much faster than the number of links even in modestly connected systems. Additionally, he knew that a single link could be part of many different loops. So, he suggested that considering loop gain elasticities would only make sense when it was possible to change one loop without influencing other ones. In his own words he asks:

> *"Under what circumstances can a set of loops have their strengths determined or changed independently by an appropriate assignment or change in the strengths of individual links in the system?"* (Kampmann 1996)

He finds a solution to that question when he analyzes the system of algebraic equations relating loop gains (g_n) and link gains ($a_{ij} = e_i$).

$$g_n = \prod_{i=1}^{q} g(e_i) \tag{14}$$

$$\ln g_n = \sum_{i=1}^{q} \ln|g(e_i)| \tag{15}$$

$$\ln|\mathbf{g}| = \mathbf{C}\ln|\mathbf{e}| \tag{16}$$

where equation (14) shows that the gain of loop n is just the product of link gains composing the loop; **g** is the vector of loops gains, where $\mathbf{g} = (g_1, g_2, \ldots, g_L)'$ with L number of loops in the system; **e** is the vector of link gains, where $\mathbf{e} = (e_1, e_2, \ldots, e_N)'$ with N number of links in the system; and, **C** is a L x N matrix.

Since there are more loops than links (L > N), the system above is over-determined. The rank ρ of the matrix is lower than L and often the system does not have a solution. A solution only exists, when the left-hand side is a linear combination of the rows of **C**. In this context, the loop gains were not independent from one another. Thus, to achieve independence it was necessary to obtain a set of loops that were linearly independent. Kampmann called this set the *independent loop set*. And, extending an important result from graph theory, he showed that in a strongly connected graph (G), with N arcs and n nodes, the number of loops in the independent loop set was N-n+1.

The impact of the independent loop set result could be potentially high, because it permits us to calculate loop elasticities from link elasticities. Unfortunately anyone using this approach must decide *which* independent loopset to use. The problem is that *any* one independent loopset will exclude certain loops (the dependent loops) from the analysis. At this point we do not know how (or whether it is possible) to use an independent loopset to calculate the elasticities of the corresponding dependent loops. Without being able to perform such a calculation, its uncertain whether practitioners can understand the information coming from Kampmann's approach in terms of the loops that are most intuitively understandable.

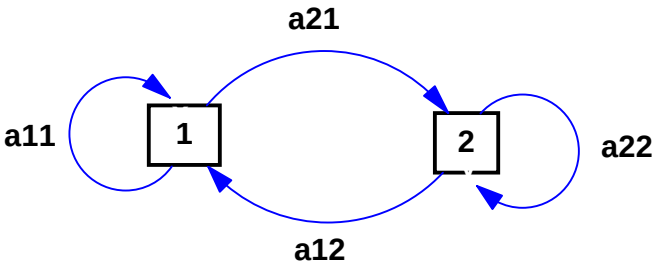
A way out of these difficulties is to sidestep the problems inherent in going from link elasticities. Kampmann (1996) provided a general formula for expressing the characteristic polynomial P(λ) in terms of the loop gains of a reduced or non-reduced

system on a theorem, that is a direct consequence of Mason's rule. His theorem stated that:

> **"Theorem:** *Let* **J** *be the n* x *n Jacobian matrix for a system with graph representation G, which may be either reduced or non-reduced, and let P(λ) be the corresponding characteristic polynomial*
>
> $$P(\lambda) = \det(\lambda I_n - \mathbf{J}) = \lambda^n + \sum_{i=1}^{n} p_i \cdot \lambda^{n-i}$$
>
> *Then* $$p_i = \sum_{D_i} (-1)^{|D_i|} g(D_i),$$
>
> *where the summation is taken over all cycle compositios D_i of order i."*

Before we provide an example illustrating the application of the theorem we introduce some notation. The *order* of a cycle is the number of state variables it contains. A *cycle composition* (D) is a collection of node-disjoint cycles that covers all nodes in G. the *gain* of a cycle composition (g(.)) is given by the product of the link gains forming the cycle composition. The order of a cycle composition (i) is the sum of the order of its constituent cycles. D_i represents all cycle compositions of order i. The *size* ($|D_i|$) of a cycle composition is the number of cycles in D.

Example 2 shows a graph theoretic diagram of a second order system in reduced form. According to the theorem above, the characteristic polynomial will have second order (n=2). The general formula for the characteristic equation of the system will be $P(\lambda) = \lambda^2 + p_1 \cdot \lambda + p_2$. The term in p_1 will be the sum of all cycles compositions (D) of order one (i=1). This corresponds to the sum of the gains of all first order loops in the diagram below. Additionally, the *size* of this cycle composition will be equal to one ($|D_1| =1$). Thus, the signs of the loop gains will be negative. The term in p_2 will be the sum of all cycles compositions (D) of order two (i=2). This corresponds to the sum of the gains of *single* second-order loops (loops involving both two state variables) and the product of gains of all *pairs* of disjoint first-order loops. The *size* of the former type of cycles will be equal to one ($|D_2| =1$) and have a negative sign. The *size* of the latter type of cycles will be equal to two ($|D_2| =2$) and have a positive sign.



$$s^2 - (a_{11} + a_{22}) \cdot s + (-a_{12} a_{21} + a_{11} \cdot a_{22})$$
$$s^2 - (g_1 + g_2) \cdot s + (-g_3 + g_1 \cdot g_2)$$

The theorem allows us obtain an implicit characterization of the loop gain elasticity. Through the application to several system dynamics models we hope to gain further insight about how the loop structure drives model behavior. Now, we apply the theorem to example one. This will illustrate the results of the theorem to a system in non-reduced form. Table 1 shows the cycle compositions, order, gains, sizes, and signs for determining the coefficients of the characteristic polynomial for example 1.

Table 1 – Cycle compositions, cycle order, cycle gains, sizes, and signs for determining the coefficients of the characteristic polynomial.

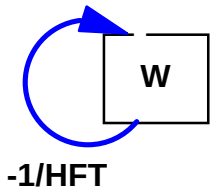
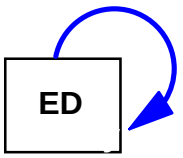
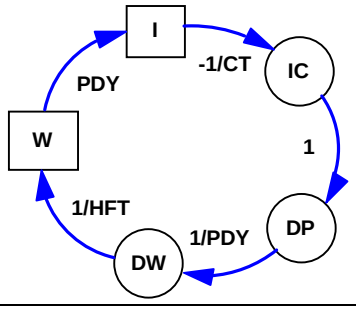
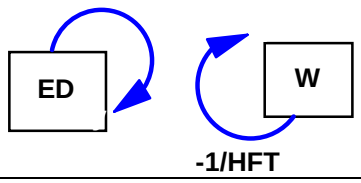
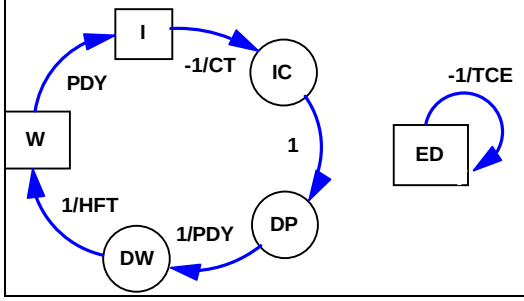
Cycle Composition (D)	**Order (*i*)**	**Gain (g(.))**	**Size ($\|D_i\|$)**	**Sign $(-1)^{\|Di\|}$**
	1	g_1 = -1/HFT	1	-1
	1	g_2 = -1/TCE	1	-1
	2	g_3= -1/(CT*HFT)	1	-1
	2	g_1 g_2= 1/(TCE*HFT)	2	+1

Table 1 – Cycle compositions, cycle order, cycle gains, sizes, and signs for determining the coefficients of the characteristic polynomial.(continued)

Cycle Composition (D)	**Order (*i*)**	**Gain (g(.))**	**Size ($\|D_i\|$)**	**Sign $(-1)^{\|D_i\|}$**
	3	$g_2\, g_3 = 1/(CT*HFT*TCE)$	2	+1

From the entries in the table above and the general formula in the theorem we obtain the terms p_1 , p_2 and p_3:

$$p_i = \sum_{D_i} (-1)^{|D_i|} g(D_i),$$

$$p_1 = - g_1 - g_2$$
$$p_2 = + g_1 g_2 - g_3$$
$$p_3 = + g_2 g_3$$

Now, it is possible to obtain the same result from the Jacobian on the reduced system. We simply compare the solution for the characteristic polynomial with the general form of the polynomial, to obtain p_1, p_2, and p_3. Where the results should be equal to the one derived above.

$$P(\lambda) = \lambda^3 + p_1 \cdot \lambda^2 + p_2 \cdot \lambda + p_3$$
$$P(\lambda) = \lambda^3 + (-g_1 - g_2)\lambda^2 + (g_1 g_2 - g_3)\lambda + g_2 g_3$$

where,

$$p_1 = \frac{1}{HFT} + \frac{1}{TCE}$$
$$p_2 = \frac{1}{HFT \cdot TCE} + \frac{1}{HFT \cdot CT}$$
$$p_3 = \frac{1}{HFT \cdot CT \cdot TCE}$$

And, the eigenvalues, for the example, in terms of the loop gains are:

$$\lambda_1 = g_2$$
$$\lambda_2 = \frac{g_1}{2} + \frac{1}{2}\sqrt{g_1^{\,2} - 4g_3}$$
$$\lambda_3 = \frac{g_1}{2} - \frac{1}{2}\sqrt{g_1^{\,2} - 4g_3}$$

Obtaining the characteristic polynomial $P(\lambda)$ in terms of the loop gains (g_n) allows us to implicitly differentiate the equation $P(\lambda; g_n) = 0$ with respect to the loop gain (g_n). From which we obtain:

$$\frac{dP(\lambda;g_n)}{dg_n} = \frac{\partial P(\lambda;g_n)}{\partial \lambda}\frac{d\lambda}{dg_n} + \frac{\partial P(\lambda;g_n)}{\partial g_n} = 0 \tag{17}$$

$$\frac{d\lambda}{dg_n} = -\frac{\partial P(\lambda;g_n)}{\partial g_n}\left(\frac{\partial P(\lambda;g_n)}{\partial \lambda}\right)^{-1} \tag{18}$$

$$E_{kn} = -\frac{\partial P(\lambda;g_n)}{\partial g_n}\left(\frac{\partial P(\lambda;g_n)}{\partial \lambda_k}\right)^{-1}\frac{\lambda_k}{\partial g_n} \tag{19}$$

3. Discussion

Forrester suggested deriving loop elasticities from link elasticities. He described how to create a system of equations involving link elasticities that in principle could be solved for loop elasticities. Unfortunately, the large number of loops relative to links meant that in practice the system will usually be over-determined. Kampmann solved this problem by introducing the idea of an independent loopset. Unfortunately there will usually not be a single independent loopset and Kampmann's approach leaves us with the problem of choosing one independent loopset from many.

We suggest sidestepping this problem by going directly to loop elasticities from Mason's rule and the characteristic equation, rather than taking the intermediate step through link elasticities. Kampmann noted that it would be possible to express the characteristic polynomial $P(\lambda)$ in terms of the loop gains (g_n) and obtain the loop gain elasticities without addressing the issue of an independent loop set. Kampmann, however, found this approach to fall short of his own standards:

> *"… the formalism [of obtaining the loop gain elasticities directly from the characteristic polynomial] is only valid if it is in fact possible, by an appropriate change to the gains of individual links in the system, to change the gains of one loop independently of all others."*

We think that Kampmann set the bar a bit too high, putting forward a standard that *no* approach can meet. Even an independent loopset can contain a loop whose gain can not be changed independently of the gains of the other loops; because even an independent loopset can contain a loop composed only of links shared with other loops in the independent loopset. Fortunately, the standard that Kampmann suggests is higher than necessary: In practice people routinely alter loop gains, even though doing so affects the gains of other loops. In fact it is unlikely that we would *ever* find a loop in the real world whose gain could vary without affecting the gains of *some* other loops.

The only relevant standard to judge the usefulness of an approach is whether it does in fact help us to improve the system by making structural (including parametric) changes. Such improvement requires an understanding of which structures in a system are most responsible for creating the behavior of concern. There is no requirement that we must find structural sources of behavior that are completely disconnected from the

rest of the system. The yardstick by which we ought to measure any analytical approach is whether it helps us in the task of understanding the connection between structure and behavior. Examining the elasticity of eigenvalues (i.e. behavior modes) with respect to loops promises to help us achieve this understand more clearly, more rigorously, and more quickly than is possible using traditional system dynamics approaches. Finding the elasticities directly, rather than indirectly, seems to us to be the best approach to finding the loop gain elasticities.

4. References

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5. Appendix

A *graph (G)* is a mathematical object consisting of *points* or *nodes(R)* and *links* or *edges (N).* We can assign properties to the elements above to expand its applications. For instance, it is possible to assign *weights* or *gains* and *directions* to links. A *weighted graph* is formed using the former assignments and a *directed graph* or *digraph* is formed using the latter. A directed edge is called an *arc*. A series of disjoint nodes (r_i) connected by arcs $r_i \rightarrow r_j \rightarrow ... \rightarrow r_q$ is called a *directed path*. Two nodes r_i and r_k are said to be *strongly connected* when there is a path from r_i to r_k and from r_k to r_i. A digraph is said to be strongly connected if for every node r_i and r_k there is a path from r_i to r_k . When the initial node (r_i) and the terminal node (r_q) in a directed path are the same ($r_i = r_q$) we

obtain a *closed path*. A closed path is called a *directed cycle* when going along the path one reaches the initial node only once. An arc connecting a node to itself ($r_i \rightarrow r_i$) is called a *self-loop.*

In a graph representation of a system dynamics model, nodes are state variables **x**(t) or auxiliary variables **y**(t) and edges are the connecting links. Arcs determine the causality assignments and link capacities provide link gains. A directed cycle maps a loop, and a self-loop is also a minor loop.