A Study of Output Curve Intersection in the Case of Delay i (i=1,2,3,......)

Jia Ren'an
Jiangxi University
17 Nanjing East Road, Nanchang
Janxi, China

Abstract

Delay function is one of the most important functions in dynamo Language in system dynamics.

This article discusses intersection characteristics of output curves in the case of delayi (i=1,2,3,......), and proves that in step-input, output curves in the case of delayi (i=1,2,3,......) doesn't intersect at the common inflection point and that when DEL1=DEL2/2=DEL3/3, the output curves in the case of delayi (i=1,2,3) doesn't intersect with each other except common initial point.

Above-mentioned result is very important in the application of delay function.

A Study of Output Curve Intersection in the Case of Delayi (i=1,2,3,......)

1. Introduction

Delay function is one of the most important functions in dynamo Language in system dynamics. The article discusses output curve intersection in the case of delay i(i=1,2,3,......).

Output curve depends on its corresponding input curve. From the first monograph on system dynamics written by the originator of system dynamics Jay W. Forrester, almost all treatises with comprehensive contents have discussed the output questions in the case of delay i(i=1,2,3,......) when its input is

$$IN(t) = \begin{cases} 0 & 0 \le t < 1 \\ a > 0 & t \ge 1 \end{cases} \quad (1)$$

Moreover, all kinds of their results are similar on the whole. Among them, the typical example is shown in Figure 1. which is given in the book "Introduction to System Dynamics Modeling with DYNAMO" (in chapter 3, item 5) written by George P. Richardson and Alexander L. Pugh III. Ever since, this figure has been quoted by many books and articles to explain output contrast in the case of delayi (i=1,2,3,......). But no article has given the intersection condition and properties of intersection point of output curve in the case of delayi (i=1,2,3,......) when its input is (1). Therefore, some people think when input is (1), its output curves must be intersected besides the intersected simulation initial point, some people even think they must be intersected at the common inflection point. Actually, this is not correct. We can give the following two theorem.

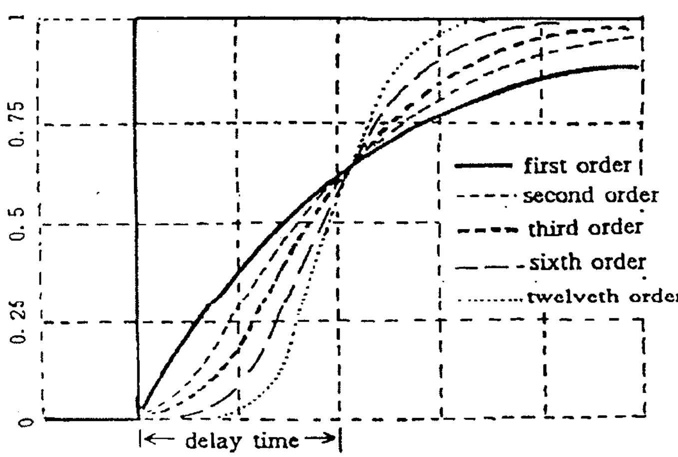

Fig. 1 First, second, third, sixth and twelfth-order delay response to step

2. Intersection Characteristics of Output Curves

Theorem 1. When input is (1), the output curves in the case of delayi (i=1,2,3,......) have no common inflection point.

Proof. Suppose the delayi (i=1,2,3) are DELAYi(IN.JK,DELi)(i=1,2,3), among them IN is (1). We can get the following proof:

(a) Output curve of DELAY1(IN.JK,DEL1)

We known subprogram

```
*    DELAY1
L    L11.K=L11.J+DT*(IN.JK-R11.JK)
N    L11=IN*DEL1                                  (2)
R    IN.KL=STEP(a,1)
R    R11.KL=L11.K/DEL1
```

which a>0 and simulation initial time is zero (following is the same). Find analytic formula of level and rate of subprogram (2).

Case 1 $0 \le t < 1$
Since

IN=0

We get

L11=0,R11=0

Case 2 $t \ge 1$
Since

IN=a>0

We get

$$\frac{dL11}{dt} = a - \frac{L11}{DEL1} \tag{3}$$

Hence we find

$$L11 = aDEL1(1 - e^{\frac{1-t}{DEL1}})$$
$$R11 = a(1 - e^{\frac{1-t}{DEL1}}) \tag{4}$$

From

$$R'11 = \frac{a}{DEL1} e^{\frac{1-t}{DEL1}} > 0$$
$$R''11 = -\frac{a}{DEL1^2} e^{\frac{1-t}{DEL1}} < 0$$

We have seen that R11 is single increase convex function, with no inflection point.

(b) Output curve of DELAY2(IN.JK,DEL2)

We known subprogram

```
*    DELAY2
L    L21.K=L21.J+DT*(IN.JK-R21.JK)
N    L21=IN*(DEL2/2)
R    IN.KL=STEP(a,1)
R    R21.KL=L21.K/(DEL2/2)                        (5)
L    L22.K=L22.J+DT*(R21.JK-R22.JK)
N    L22=L21
R    R22.KL=L22.K/(DEL2/2)
```

Find analytic formula of level and rate of subprogram (5).

Case 1 $0 \le t < 1$

Since

IN=0

We get

L21=0,L22=0,R21=0,R22=0

Case 2 $t \geq 1$

Since

IN=a>0

We can get

$$\frac{dL21}{dt} = a - \frac{L21}{(DEL2/2)} \tag{6}$$

$$L21 = a(DEL2/2)(1 - e^{\frac{1-t}{DEL2/2}})$$

$$R21 = a(1 - e^{\frac{1-t}{DEL2/2}}) \tag{7}$$

From

$$\frac{dL22}{dt} + \frac{L22}{DEL2/2} = a(1 - e^{\frac{1-t}{DEL2/2}})$$

We find

$$L22 = aDEL2/2[1 - (1 - \frac{1}{DEL2/2} + \frac{t}{DEL2/2})e^{\frac{1-t}{DEL2/2}}] \tag{8}$$

$$R22 = a[1 - (1 - \frac{1}{DEL2/2} + \frac{t}{DEL2/2}e^{\frac{1-t}{DEL2/2}}] \tag{9}$$

From

$$R'22 = \frac{a}{(\frac{DEL2}{2})^2} e^{\frac{1-t}{DEL2/2}}(t-1) > 0 \tag{10}$$

$$R''22 = \frac{a}{(\frac{DEL2}{2})^3} e^{\frac{1-t}{DEL2/2}}(1 + \frac{DEL2}{2} - t) \tag{11}$$

We have seen that

for $1 \leq t < 1 + DEL2/2$ from R"22>0, R22 is concave upwards

for t>1+DEL2/2 from R"22<0, R22 is convex upwards

for t=1+DEL2/2 R"22=0 and R22=a$(1-\frac{2}{e})$

Therefore the inflection point of R22 is $(1+\frac{DEL2}{2}, a(1-\frac{2}{e}))$.

(c) Output curve of DELAY3(IN.JK,DEL3)

We known subprogram

```
*    DELAY3
L    L31.K=L31.J+DT*(IN.JK-R31.JK)
N    L31=IN*DEL3/3
R    IN.KL=STEP(a,1)
R    R31.KL=L31.K/(DEL3/3)
L    L32.K=L32.J+DT*(R31.JK-R32.JK)                    (12)
N    L32=L31
R    R32.KL=L32.K/(DEL3/3)
L    L33.K=L33.J+DT*(R32.JK-R33.JK)
```

N L33=L32
R R33.KL=L33.K/(DEL3/3)

Now, let's find the analytic formula of level and rate of subprogram (12).

Case 1 $0 \le t < 1$
Since
IN=0
We get
L3i=0 (i=1,2,3)
R3i=0 (i=1,2,3)

Case 2 $t \ge 1$
Since
IN=a>0
We have the following result:

From
$$\frac{dL31}{dt} = a - \frac{L31}{DEL3/3}$$

We can get
$$L31 = aDEL3/3(1 - e^{\frac{1-t}{DEL3/3}})$$
$$R31 = a(1 - e^{\frac{1-t}{DEL3/3}}) \tag{13}$$

From
$$\frac{dL32}{dt} + \frac{L32}{DEL3/3} = a(1 - e^{\frac{1-t}{DEL3/3}})$$

We can get
$$L32 = aDEL3/3[1 - (1 - \frac{1}{DEL3/3} + \frac{t}{DEL3/3})e^{\frac{1-t}{DEL3/3}}] \tag{14}$$

$$R32 = a[1 - (1 - \frac{1}{DEL3/3} + \frac{t}{DEL3/3})e^{\frac{1-t}{DEL3/3}}] \tag{15}$$

From
$$\frac{dL33}{dt} + \frac{L33}{DEL3/3} = a[1 - (1 - \frac{1}{DEL3/3} + \frac{t}{DEL3/3})e^{\frac{1-t}{DEL3/3}}]$$

We find
$$L33 = aDEL3/3[1 - (1 - \frac{1}{DEL3/3} + \frac{1}{2(DEL3/3)^2} + \frac{t}{DEL3/3} - \frac{t}{(DEL3/3)^2} + \frac{t^2}{2(DEL3/3)^2}e^{\frac{1-t}{DEL3/3}}] \tag{16}$$

$$R33 = a[1 - (1 - \frac{1}{DEL3/3} + \frac{1}{2(DEL3/3)^2} + \frac{t}{DEL3/3} - \frac{t}{(DEL3/3)^2} + \frac{t^2}{2(DEL3/3)^2})e^{\frac{1-t}{DEL3/3}}] \tag{17}$$

From

$$R'33 = \frac{a}{2(DEL3/3)^3} e^{\frac{1-t}{DEL3/3}} (1-t)^2 > 0 \qquad (18)$$
$$R''33 = \frac{a}{2(DEL3/3)^3} (1-t) e^{\frac{1-t}{DEL3/3}} \frac{t-(1+2DEL3/3)}{DEL3/3}$$

We have seen that

for $1 \le t < 1 + \frac{2DEL3}{3}$ from R"33>0, R33 is concave upwards

for $t > 1 + \frac{2DEL3}{3}$ from R"33>0, R33 is convex upwards

for $t = 1 + \frac{2DEL3}{3}$ R"33=0 and R33= $a(1 - \frac{5}{e^2})$

Therefore the inflection point of R33 is $(1 + 2DEL3/3, a(1 - \frac{5}{e^2}))$.

To sum up, when t>1, the corresponding formulas of output curves in the case of delayi(in.jk,deli)(i=1,2,3) are:

R11 is single function with no inflection point; the ordinate of inflection point of R22, R33 have nothing to do with delay time, and the difference of ordinates is

$a(1\text{-}2/e)\text{-}a(1\text{-}5/e^2)=(5\text{-}2e)/e^2$

So analytic output curves in the case of delayi(i=1,2,3,......) have no common inflection point.

If we simulate above-mentioned subprograms in the case of delayi(i=1,2,3,......) in computer, every result is similar with above-mentioned corresponding analytic curve. But when simulation step is infinitely small, its accuracy can be small enough. So the conclusion of computer simulation result is identical with the above-mentioned analytic proof.

This theorem proves that output curves in the case of delayi(i=1,2,3,......) can't be intersected at the common inflection point when its input is (1).

Q.E.D.

Theorem 2. Output curves in the case of delayi(i=1,2,3) don't be intersected again except common initial point (1,0) when its input is (1), and DELI=DEL2/2=DEL3/3.

Proof. Let R11=R22

$$a(1 - e^{\frac{1-t}{DEL1}}) = a[1 - (1 - \frac{1}{DEL2/2} + \frac{t}{DEL2/2} e^{\frac{1-t}{DEL2/2}}]$$

There is only one solution t=1

Let R22=R33

$$a[1 - (1 - \frac{1}{DEL2/2} + \frac{t}{DEL2/2}) e^{\frac{1-t}{DEL2/2}}] = a[1 - (1 - \frac{1}{DEL3/3} + \frac{t}{DEL3/3} + \frac{1}{2(DEL3/3)^2} - \frac{t}{(DEL3/3)^2} + \frac{t^2}{2(DEL3/3)^2}) e^{\frac{1-t}{DEL3/3}}]$$

There is only duplicate solution $t_{1,2}=1$

Q.E.D.

Example 1. Suppose corresponding programs in the case of delayi(i=1,2,3) as follows respectively:

```
*
R       R11.KL=DELAYI(IN.JK,DELI)
R       DELI=2
R       IN.KL=STEP(5,1)
NOTE
L       L21.K=L21.J+DT*(IN.JK-R21.JK)
N       L21=IN*(DEL2/2)
R       R21.KL=L21.K/(DEL2/2)
L       L22.K=L22.J+DT*(R21.JK-R22.JK)
N       L22=L21
R       R22.KL=L22.K/(DEL2/2)
C       DEL2=4
NOTE
R       R33.KL=DELAY3(IN.JK,DEL3)
C       DEL3=6
NOTE
SPEC    DT=.125/LENGTH=12/PLTPER=.125
PLOT    IN,R11,R22,R33
OPT     PR
```

Please simulate the program in computer and contrast the results of output curves.

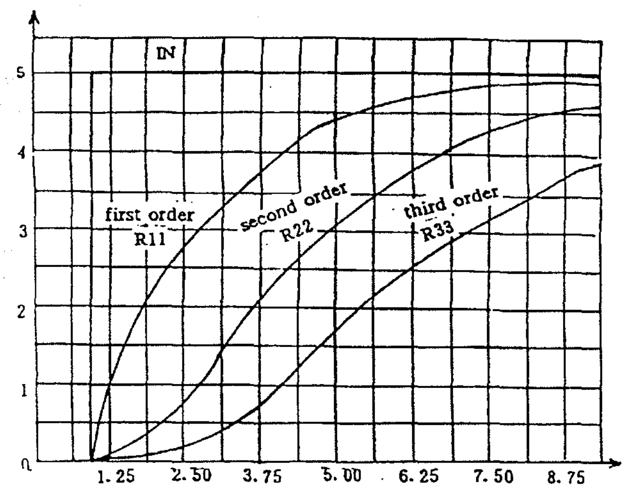

Fig. 2. Contrast of output in the case of delayi(1,2,3)

Solution. The output curves are shown by Figure 2. after the simulation.

Correctness of above-mentioned theorem is further proved by example 1.

References

George P. Richardson and Alexander L. Pugh III, Introduction to System Dynamics Modelling with DYNAMO, 1981.
Jay W. Forrester, Principles of Systems, The MIT Press, 1968.
Jay W. Forrester, Industrial Dynamics, Cambridge, Mass., The MIT Press, 1961.
Jia Ren'an, A Course in System Dynamics, The Jiangxi People's Press, 1992.
Wang Qifan, System Dynamics, Qing hua University Press, 1985.