Controlling primary income distribution and employment under increasing returns[♥]

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Abstract. This paper demonstrates that the Kennedy – Goodwin macroeconomic model of capital accumulation (KGM) does not reflect direct increasing return. The author presents its two versions: KGM-I with weakening roundabout increasing return and KGM-II with reinforcing roundabout increasing return. Both have the common intensive form and same asymptotically stable stationary state. KGM-II is changed to allow for direct increasing return, whereby the growth rate of employment ratio positively influences the growth rate of labour productivity. If the latter effect is strong enough, the dynamic stationary state is locally repelling and bifurcates into closed orbits. Their period is estimated. This paper supposes a closed loop control that stabilizes the oscillatory dynamics of the main macroeconomic variables, maintaining profitability and employment under direct and reinforcing roundabout increasing returns. It is proved that the supposed policy would be destabilizing if the direct scale effect were powerfully negative that is not empirically correct. Simulation runs confirm analytical findings. This paper yields insights for public debate on competent pro-growth stabilization policy.

Key words: growth cycle, primary income distribution, employment, stabilization policy.

Introduction

According to P. Krugman, three most important elements for the economy overall are productivity, income distribution and unemployment (Krugman 1990: 7). The missing of increasing returns (economies of scale) among ‘the most important elements’ is not surprising since neoclassical growth theory has never been very comfortable with them (Kaldor 1981).

“The economics profession for a long time tended to minimize the importance of the economies of scale (von Weizsäcker 1993: 242).” The Nobel prize winner K. Arrow has expressed the critical thought as well: “Though largely ignored in university economics (except in isolated courses like industrial organization), increasing returns and the associated non-perfect competition and economics of specialization are ubiquitous and very important in real economy…What are the implications of this for certain crucial macroeconomic problems…?" (Increasing returns ….1998: xix).

There has been a long tradition, going back to Adam Smith (1776), that division of labour and technological progress are somehow intrinsically associated with increasing returns. Still

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there is substantial ambiguity in notion of economy of scale (or increasing return as the synonym) in the macroeconomic literature.

In the past, only Harrod–neutral (or purely labour-augmenting) technological change could be introduced into neoclassical growth without leading to bizarre results (Nordhhaus 1973, Jones 2003). The implied smooth transition to stationary growth path yet contradicts empirical evidence of non-constancy of the capital income share, economic fluctuations of different types. The fine properties of neoclassical balanced growth usually do not survive if the initial premises are relaxed.

The traditional neoclassical approach assumes that the distribution of income is fixed by the profit-maximizing behaviour of firms and relies on production factors' perfect substitution to attain the static or dynamic stationary state. In this case monotonic convergence to stationary state is guaranteed for explicit or implicit negative returns to scale although the adjustment is usually slow.[1]

Neoclassical growth models were "saved" from such restrictiveness by the introduction of the conception of induced innovation. It was argued that it has an important advantage over the marginal productivity theory since it is dynamic in the sense of relying on innovation possibilities rather than on the characteristics of a static production function (Kennedy 1964).

Its other surmised advantage is the implicit postulating of bounded rationality. The firms maximize the reduction in unit costs subject to technical change (or invention possibility) frontier. These firms as assumed do not anticipate changes in the real wage or the real cost of capital, r. Moreover, maximization of profits is viewed unrealistic, since the non-linear technical progress function is, in general, not integrable.

Under the usual neoclassical assumptions on income distribution among labour and capital according to these factors imaginary marginal productivities, and in addition when the innovation possibility curve takes the concave form assumed by Kennedy and Samuelson, the economy settles down into a balanced-growth path exactly like that of the purely labour-augmenting case (Samuelson 1965; Drandakis and Phelps 1966).

This conception won popularity and even appeared as the basis for a study of class conflict (Bowles, Kendrick 1970: 196 – 200). Still the persistent unemployment and growth cycles were mostly ignored. The neoclassical conclusions on asymptotically stable balanced-growth path were challenged by a pioneer work (Goodwin 1972) that defined a model for the share of labour in national income and the employment ratio in a highly stylised capitalist economy. This model generates a path of development where growth and cycles appear simultaneously without having distinct causes. The very notion of cyclical growth (or growth cycle) is a profound synthesis indeed.

In Goodwin's model the interaction of income distribution with capital accumulation generates business cycles. The model written in continuous time belongs to the class of model of the Lotka – Volterra type. The employment ratio serves as the prey, while the wage share in net national product (NNP) acts as a predator. In order to obtain reasonable economic solutions, it is necessary to put some restrictions on the parameters. Income distribution is

[1] It is proved, in particular, that an implicit requirement of non-realistic dominance of negative returns to scale is necessary and sufficient for saving the property of asymptotical stability of stationary state if the condition of full employment is relaxed under other typical neoclassical conditions (Ryzhenkov 2005b, 2006). This neoclassical requirement corresponds to a popular belief that slower increases or decreases in labour productivity promote employment at the macroeconomic level. See also Section 2.1.

determined by the dynamics of real wages and productivity. Exogenous technical progress is one of the main weaknesses of Goodwin's model.

The Kennedy–Goodwin model (KGM-I) was elaborated in (Shah, Desai 1981; Ploeg 1983; Ploeg 1987). Allowing entrepreneurs to choose the cost-minimising direction of technological progress extended Goodwin's basic predator – prey model. It was claimed, that this extra weapon gave firms the power to eventually eliminate the class struggle as the introduction of induced technical progress changes the stability properties of the model.

As the Goodwin model is an economic analogue of the Lotka – Volterra predator – prey model, its stationary state is a centre around which the economy moves in closed orbits whose length is determined by initial conditions. This result is due to the implicit assumption that each side in the class struggle has only one weapon – workers can bargain on strength of full employment and capitalists can determine the growth of employment by their investment decisions.

Then capitalists have gotten one more weapon – choice of the induced rate of technical change. This has the effect that instead of being perpetually cyclical around the steady state, owing to firms optimising behaviour the economy becomes locally stable around the stationary state.

Still it shares with the conception of induced innovation the same weaknesses: the inability to take proper account of increasing returns. This is consequence of a simplistic presentation of technology opportunities frontier and of production relations (see below).

The first objective of this paper is to provide an alternative to KGM-I, which can generate everlasting growth cycles. In order to achieve this purpose, the basic model is changed to allow for direct increasing return, whereby the growth rate of employment ratio positively influences the growth rate of labour productivity. If the latter effect is strong enough, the dynamic stationary state is locally repelling and bifurcates into closed orbits. This result is to be established in this paper with the aid of the Andronov – Hopf bifurcation theorem, which was used to prove the existence of closed orbits in a different context (see Ploeg 1987, Ryzhenkov 2000, 2006 among others).

The widely held view of social science is that oscillatory macroeconomic systems are usually undesirable because ups and especially downs bring about the wastage of human and other resources. I agree with the opinion (Yellen, Akerlof 2006: 19) that "…stabilization policy reduces average levels of joblessness and raises average output by a nontrivial amount. Stabilization policy also raises social welfare if, as seems likely, welfare deteriorates nonlinearly with increases in unemployment." Smoothing or eradicating oscillations requires more structural than numerical changes.

As shown in (Franco 1990; Ryzhenkov 2005a, 2005b, 2006), design of effective policies to control oscillations is a problem that goes beyond the Classical Optimal Control Theory of non-linear systems, and it belongs to the Structural Control Theory. Its application and development allows conceiving a policy of primarily income distribution that stabilizes the oscillatory dynamics of the main macroeconomic variables, sustaining total profit and employment.

The present author agrees with a view (Milling 1990: 33): "Policy Support Systems should work as a kind of "transitional object" that allows experimentation, development and the analysis of different scenarios into the future. They should lead to a better understanding of the structure and the behaviour of complex business organizations." This paper elaborates a control law of capital accumulation and pro-growth stabilization policy.

Section 1 presents the Kennedy – Goodwin model; Section 2 offers the generalised model that includes reinforcing direct and roundabout increasing returns; Section 3 synthesises a control law for the oscillatory macroeconomic system; the Appendix contains propositions on bifurcations giving birth to growth cycles and formal proofs.

1 The Kennedy – Goodwin model (KGM-I)

1.1 The model assumptions, structure and equations

The model is formulated in continuous time. Time derivatives are denoted by a dot, while a hat will indicate growth rates. Let q, k, l, n, w, $v = l/n$, $a = q/l$, $\sigma = k/q$ and $u = wl/q$ denote real output, capital, employment, labour supply, the real wage rate, employment rate, labour productivity, capital–output ratio and the share of wages in the national income, respectively. All variables are in real terms and net of depreciation.

It is assumed that capitalists invest all profit or its part in fixed capital without material delay, all wages are consumed, the product market is always in equilibrium and all savings serve as internal finance for investment purposes.

The incremental fixed capital equals net investment (savings) in the following equation

$$\dot{k} = x(1-u)q, \tag{1}$$

where x is the capitalist saving (investment) ratio, $0 < x \leq 1$.

The dynamics of the real wage follow from the wage-bargaining equation

$$\hat{w} = -\gamma + \rho v, \gamma > 0, \rho > 0\,. \tag{2}$$

A reduction in the reserve army of unemployed facilitates workers' bargaining strength and therefore promotes the growth in real wages.

The equation for the time derivative of capital–output ratio is

$$\dot{\sigma} = -\psi(u)\sigma, \psi'(u) < 0\,.^{2} \tag{3}$$

The interpretation of (3) is based on the hypothesis that firms maximize the reduction in unit costs

$$R = -\frac{\partial\left(\dfrac{wl+rk}{q}\right)}{\partial t} = -\frac{\partial\left(\dfrac{w}{a}+r\sigma\right)}{\partial t} = -\left(-\frac{w\dot{a}}{aa}+r\dot{\sigma}\right) = u\frac{\dot{a}}{a} - r\dot{\sigma}$$
$$= \alpha u + \eta(1-u)\,, \tag{4}$$

where $r = (1-u)/\sigma$ is the profit rate,[3] $\alpha \equiv \hat{a}$ is the growth rate of labour productivity governed by (5)

$$\alpha = \phi[\eta(t)], \quad \phi' < 0,\ \phi'' < 0, \tag{5}$$

where, in its turn, the growth rate of the output–capital ratio is

$$\eta(t) \equiv -\hat{\sigma}. \tag{6}$$

Differentiating (4) with respect to η yields $R' = u\phi' + (1-u) = 0$ and $R'' = u\phi'' < 0$. These are the necessary and sufficient conditions for maximal reduction in unit cost. In particular, at the stationary state $R' = u_a\phi'(0) + (1-u_a) = 0$ and $1 > u_a = \dfrac{1}{1-\phi'(0)} > 0$. Correspondingly, the cost–minimizing direction of technical innovation is given by (3) where

$$\psi(u) = \eta = -\hat{\sigma} = (\phi')^{-1}[(u-1)/u],\ \psi'(u) = \frac{1}{\phi''}\left(1-\frac{1}{u}\right)' = \frac{1}{\phi'' u^{2}} < 0\,.$$

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[2] The sign of the derivative is derived below.

[3] KGM-I assumes that profit rate is identical to cost of capital for the firms management. In my view, this strong assumption deserves a critical reconsideration. This is beyond the scope of my present paper.

This explains the negative sign of the derivative in the equation (3).

The rate of change of the employment ratio is given by

$$\hat{v} = \hat{k} - \hat{\sigma} - \hat{a} - \beta$$
$$= x\frac{1-u}{s} + \psi(u) - \phi[\psi(u)] - \beta, \qquad (7)$$

where β = const is the growth rate of labour force.

The equation (8) defines the rate of change of the relative wage

$$\hat{u} = \hat{w} - \hat{a}$$
$$= -\gamma + \rho v - \phi[\psi(u)]. \qquad (8)$$

The Figure 1a represents the detailed KGM-I causal-loop structure. The stock and flow variables are joined by arrows. An arrow, going from one variable to another, means that the latter variable is a function of the former. The arrows, differently shaped, represent either integration of the flows into stocks or information linkages. The principle of secular and cumulative causation, which is realised in this and other analogous figures, prompts the following matter-of-fact interpretation of increasing returns in this paper.

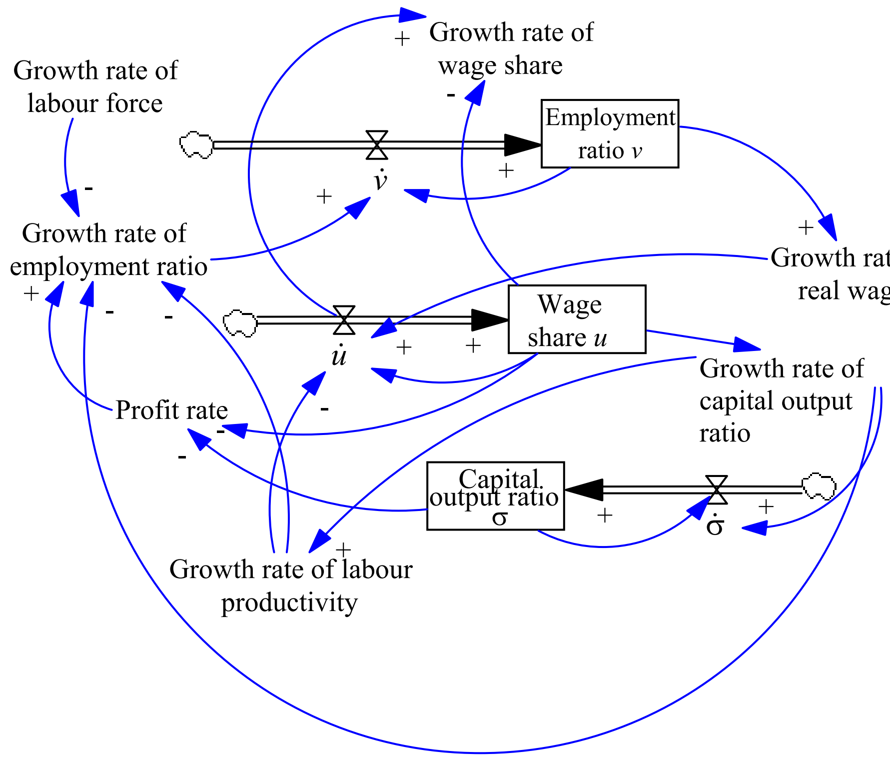

Figure 1a The causal structure of KGM-I

In my view, *direct economy of scale* (*direct increasing return*) manifests itself in a positive partial derivative of growth rate of labour productivity ($\hat{a}$) with respect to employment ratio (v) or growth rate of employment ratio ($\hat{v}$): $\frac{\partial \hat{a}}{\partial v} > 0$ or $\frac{\partial \hat{a}}{\partial \hat{v}} > 0$. *Roundabout economy of scale* (*roundabout increasing return*) manifests itself in a positive partial derivative of growth rate of labour productivity with respect to employment ratio (v) or growth rate of employment ratio ($\hat{v}$) intermediated by other variable or variables (x_i): $\frac{\partial \hat{a}}{\partial x_1}\frac{\partial x_1}{\partial v} > 0$ or $\frac{\partial \hat{a}}{\partial x_1}\ldots\frac{\partial x_i}{\partial v} > 0$, $\frac{\partial \hat{a}}{\partial x_1}\frac{\partial x_1}{\partial \hat{v}} > 0$ or $\frac{\partial \hat{a}}{\partial x_1}\ldots\frac{\partial x_i}{\partial \hat{v}} > 0$, $i \in \{2,\ldots,I\}$.

Economy of scale (increasing return) is *reinforcing* if a *positive* feedback loop connects the growth rate of labour productivity with employment ratio and (or) its growth rate. Economy of scale (increasing return) is *weakening* if a *negative* feedback loop connects the growth rate of labour productivity with employment ratio and (or) its growth rate.

The intensive deterministic form of KGM-I for the initial system of economic growth, the class struggle and induced technical change is (3), (9), (10) (the equation (9) follows from the equation (7), the equation (10) – from (8)).

$$\dot{v} = [x\frac{1-u}{\sigma} + \psi(u) - \phi(\psi(u)) - \beta]v, \tag{9}$$

$$\dot{u} = [-\gamma + \rho v - \phi(\psi(u))]u. \tag{10}$$

At a fixed-point ($\dot{\sigma}$, $\dot{v}$, $\dot{u}$) = 0.

The Harrod – Domar condition gives the non-trivial stationary state of this system $E_a = (\sigma_a, v_a, u_a)$:

$$\sigma_a = x(1-u_a)/(\alpha_a + \beta),\ v_a = (\gamma + \alpha_a)/\rho,$$

$$1 > u_a = \psi^{-1}(0) = \frac{1}{1-\phi'(0)} > 0; \tag{11}$$

the stationary growth rates of output-capital ratio, labour productivity and net output are $\eta_a = 0,\ \alpha_a = \phi(0),\ d = x\frac{1-u_a}{\sigma_a} = \alpha_a + \beta > 0,$ correspondingly.

1.2 KGM-I in a non-linear co-operative-competitive network

Let G be an open subset of R^n. A differential equation

$$\dot{x} = f(x) \tag{12}$$

defined on $G \subseteq R^n$ is *co-operative* if $\partial f_i(x)/\partial x_j \geq 0$ for all $x \in G$ and all $i \neq j$ (the growth of every component is enhanced by an increase in any other component). *Competitive* systems are defined by $f_i(x)/\partial x_j \leq 0$ for all $i \neq j$ (Hofbauer, Sigmund 1988: 158). The system (3), (9), (10) represents a mixed case, i.e., a *competitive-co-operative* system in the three-dimensional phase space:

$$\partial f_1(\sigma)/\partial v = 0,\ \partial f_1(\sigma)/\partial u = -\psi'(u)\sigma > 0,$$

$$\partial f_2(v)/\partial \sigma = -x\frac{1-u}{\sigma^2}v < 0,\ \partial f_2(v)/\partial u = \{\psi'(u) - \phi'[\psi(u)]\psi'(u) - x\frac{1}{\sigma}\}v < 0,$$

$$\partial f_3(u)/\partial \sigma = 0,\quad \partial f_3(u)/\partial v = \rho u > 0.$$

The partial derivatives of the system (3), (9), (10) are elements of the Jacoby matrix in the equation (13). This matrix corresponds to the causal-loop structure of the system (3), (9), (10) represented by Figures 2–6.

$J =$

$\hat{\sigma}$	0	$-\psi'(u)\sigma > 0$
$-x\frac{1-u}{\sigma^2}v < 0$	$\hat{v}$	$\{\psi'(u) - \phi'[\psi(u)]\psi'(u) - x\frac{1}{\sigma}\}v < 0$
0	$\rho u > 0$	$\hat{u} - \phi'[\psi(u)]\psi'(u)u$

. (13)

Notice that the wage share (the main "predator") adversely affects growth of the employment ratio (the prey) and it activates growth of the capital–output ratio. This ratio (the second "predator") inhibits the growth of the employment ratio. The rising employment ratio is promoting the growth of the wage share. The system has fairly satisfactory self-regulating properties in vicinity of the stationary state (11) under above restrictions on its functions.

It becomes clear from the inspecting of the Figures 2–6 that KGM-I has two main and three little feedback loops. The initial main loop connects the wage share and the employment ratio, the additional main loop of a higher order of complexity – the relative wage, employment ratio and capital–output ratio. The little loops involve σ, v and u individually. Integration of $\dot{\sigma}$, $\dot{v}$, $\dot{u}$ creates the implicit delays fostering fluctuations. As the partial derivatives $\partial f_i / \partial f_i$, $i = 1, 2, 3$, change their signs, the polarity of each little feedback loop alters. Because of alternating polarity the little feedback loops can be balancing or reinforcing.

The forms of proportional and derivative feedback control, used in the model economy, are not sufficient to eliminate deviations from the stationary state entirely and tend typically to cause converging fluctuations. The next section explains rigorously local stability of the non-trivial stationary state (11).

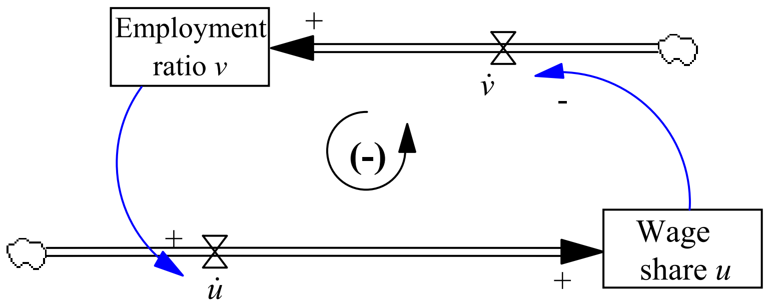

Figure 2 The first negative loop of KGM-I

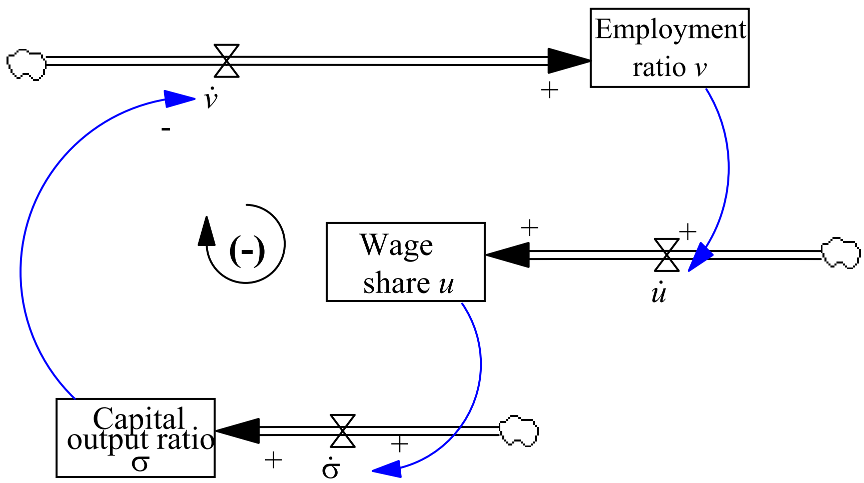

Figure 3 The (additional) second negative loop of KGM-I

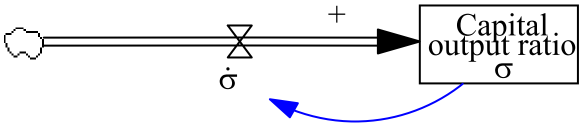

Figure 4 The first loop of KGM-I with alternating polarity determined by $\mathrm{sign}[-\psi(u)]$

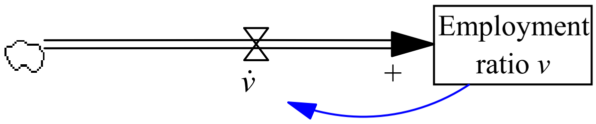

Figure 5 The second loop of KGM-I with alternating polarity determined by $\mathrm{sign}[x\frac{1-u}{\sigma}+\psi(u)-\phi[\psi(u)]-\beta]$

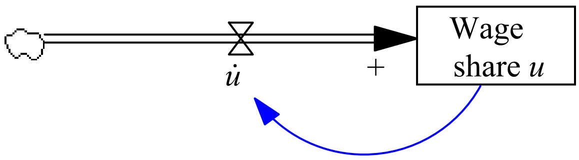

Figure 6 The third loop of KGM-I with alternating polarity determined by: $\mathrm{sign}\{-\gamma+\rho v-\phi[\psi(u)]-\phi'[\psi(u)]\psi'(u)u\}$

1.3 Local stability of the non-trivial stationary state in KGM-I

Consider the local behaviour of the solution of the unforced continuous–time non-linear system (3), (9), (10) represented as (14) near the stationary state $E_a=(\sigma_a, v_a, u_a)$ in R^3. The linear equation

$$\dot{y} = J_a y, \qquad (14)$$

where J_a is the Jacoby matrix, can be solved explicitly. The equation (15) defines the Jacoby matrix for the stationary state $E_a=(\sigma_a, v_a, u_a)$:

$$J_a = \begin{array}{|c|c|c|} \hline 0 & 0 & -\psi'(u_a)\sigma_a \\ \hline -x\dfrac{1-u_a}{\sigma_a{}^2}v_a & 0 & \{\psi'(u_a)-\phi'(0)\psi'(u_a)-\dfrac{x}{\sigma_a}\}\,v_a \\ \hline 0 & \rho u_a & -\phi'(0)\psi'(u_a)u_a \\ \hline \end{array} \quad . \qquad (15)$$

The convexity and shape of the technical change frontier (5) determines the derivative of (3), $\psi'(u_a) = \frac{[1-\phi'(0)]^2}{\phi''} < 0$, and is crucial for the local stability properties of stationary state in this model.

The characteristic equation is

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0. \tag{16}$$

The Routh – Hourwitz necessary and sufficient conditions for the local stability are:

$$a_0 > 0, \tag{C-1}$$
$$a_1 > 0 \tag{C-2}$$

and

$$a_1 a_2 > a_0. \tag{C-3}$$

All these conditions are satisfied (Shah, Desai 1981: 1007) because

$a_0 = -det(J) = -\rho u_a\, dv_a\, \psi'(u_a) > 0$; $a_1 = -[\psi'(u_a)/u_a - x\frac{1}{\sigma_a}]\, v_a\, \rho u_a > 0$;

$a_2 = -trace(J) = -[-\phi'(0)\psi'(u_a)u_a] > 0$; and, finally, $a_1 a_2 > a_0$.

1.4 Roundabout increasing return and its weakening in KGM-I

Direct scale effect is obviously absent in KGM-I. It is necessary to figure out whether this original model has any roundabout increasing return either reinforcing or weakening.

Although the Figure 1a is sufficient for this undertaking, a more detailed presentation below is helpful. At first, I present the all feedback loops containing growth rates of labour productivity, employment ratio as well as employment ratio itself. Secondly, I determine the respective loops' polarity.[4]

There are two negative feedback loops for the rate of change of labour productivity:

loop Number 1 of length 3 – negative: $\hat{a} \xrightarrow{-} \dot{u} \rightarrow u \rightarrow \hat{s} \rightarrow \hat{a}$;

loop Number 2 of length 7 – also negative: $\hat{a} \xrightarrow{-} \hat{v} \rightarrow \dot{v} \rightarrow v \rightarrow \hat{w} \rightarrow \dot{u} \rightarrow u \rightarrow \hat{s} \rightarrow \hat{a}$.

Loop No. 2 reveals the presence of roundabout increasing return only while direct scale effect is absent. This roundabout increasing return is weakening as it affects negatively the origin for itself – the growth rate of employment ratio. This property is, according to (Landmann 2004: 34), in striking contrast to reality as "employment and productivity are strongly and positively correlated over the business cycle as both variables fluctuate in a robustly procyclical way."

There are three additional negative feedback loops for the rate of change of employment ratio:

loop Number 1 of length 6 – negative: $\hat{v} \rightarrow \dot{v} \rightarrow v \rightarrow \hat{w} \rightarrow \dot{u} \rightarrow u \rightarrow \hat{s} \xrightarrow{-} \hat{v}$;

loop Number 2 of length 6 – negative: $\hat{v} \rightarrow \dot{v} \rightarrow v \rightarrow \hat{w} \rightarrow \dot{u} \rightarrow u \xrightarrow{-} \frac{1-u}{s} \rightarrow \hat{v}$;

[4] The arrows below are marked explicitly only by the negative sign for negative partial derivatives, while positive partial derivatives are implied by arrows without sign.

loop Number 4 of length 9 – also negative:

$$\hat{v} \to \dot{v} \to v \to \hat{w} \to \dot{u} \to u \to \hat{s} \to \dot{s} \to s \xrightarrow{-} \frac{1-u}{s} \to \hat{v}.$$

All loops including employment ratio (v), except the loop presented on the Figure 5, are already included in the above loops for $\hat{v}$. All these loops are negative. There is no reinforcing increasing return in KGM-I, because this model implies a trade-off between employment and labour productivity.

1.5 The Reality Checks of KGM-I

Taking the US official statistics on NNP and fixed assets in current prices and the number of employees for 1948–2002, I have linearized and tested the basic equation (5) of KGM-I empirically by the OLS method. Its probabilistic form results from adding disturbances in the equations for growth rates on the right hand side. More complicated probabilistic versions outside the scope of this paper would take into account measurement errors as well.

An equation of a linear regression has the standard form

$$y_t = b_0 + b_1 z_t + e_t \quad (t = t_0, t_0 + 1, t_0 + 2, \ldots, t_0 + m),$$

where y_t is the t^{th} observation of the dependent variable, z_t is the t^{th} observation of the independent (explanatory) variable, e_t is t^{th} unobserved disturbance term; b_0 and b_1 denote the constant term and the regression coefficient to be estimated, m +1 is the number of observations.

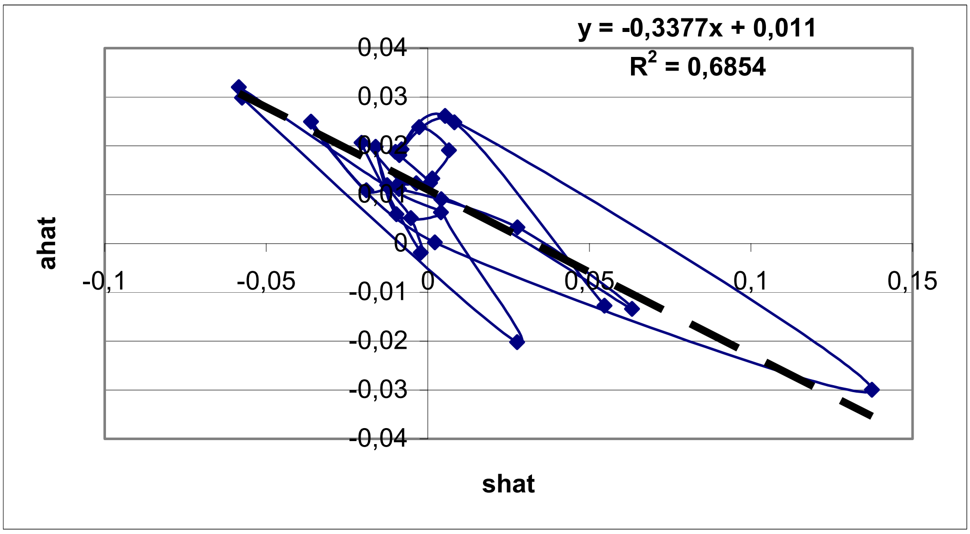

Figure 7 The OLS estimate for linearized equation (5) for growth rates of capital-output ratio (shat ≡ $\hat{\sigma}$) and labour productivity (ahat ≡ $\hat{a}$), the USA, 1970–2000

If the linearized equation (5) were valid, there would be a positive correlation between growth rate of labour productivity and that of capital-output ratio as random variables. In fact, the correlation coefficient and regression coefficient (the slope) are negative, contrary to the postulating by KGM-I (Figure 7). If instead of using 1970–2000, one takes 1949–2002 the estimates for the same correlation and regression coefficients are again negative (Table 1).[5]

[5] Searching for parameters' confidence intervals according to the standard econometric procedures is meaningless because of enormous specification errors in KGM-I. See a critique of

Table 1. The OLS estimations for the linearized equation (5)

	1949–2002	1970–2000
b_0	0.017	0.011
b_1	–0.424	–0.338
R^2	0.470	0.685
R	–0.685	–0.828

In my view, the calculated negative correlation between the growth rates of capital-output ratio and labour productivity is spurious. This is brought about by neglect of economies of scale (see correction of the equation (5) in Section 2.1 below).

The next task is to show that the modified KGM-I includes reinforcing positive returns *in embryo*. An alteration of KGM-I allows for reinforcing roundabout increasing return in KGM-II in the next section.

1.6 KGM-II (with reinforcing roundabout increasing return)

Now we add the capital intensity as the model explicit variable. The rate of change of labour productivity is determined by a Kaldorian technical progress function $\Omega(\hat{k}-\hat{l})$ in the equivalent form:

$$\hat{a}=\phi(\eta)=\Omega(\widehat{k/l}),\ \ \phi^{-1}(\hat{a})=\eta=\hat{a}-(\hat{k}-\hat{l})=\hat{a}-\Omega^{-1}(\hat{a}). \tag{5a}$$

The growth rate of capital-output ratio becomes

$$\hat{\sigma}=\widehat{k/l}-\hat{a}. \tag{6a}$$

The partial derivative of the growth rate of capital intensity with respect to wage share is positive as

$$\frac{\partial\widehat{k/l}}{\partial u}=\frac{\partial(\hat{a}+\hat{\sigma})}{\partial u}=\frac{\partial\phi[\psi(u)]}{\partial u}+\frac{\partial\hat{\sigma}}{\partial u}=\phi'\ \psi'(u)-\psi'(u)>0.$$

After these changes, the revised KGM-I becomes KGM-II. This revision produces no effect on their common intensive form represented by the equations (3), (9) and (10) and on non-trivial stationary state defined by the equation (11).

The Figure 1b shows the structure of the modified KGM-II. It is necessary again at first to portray the all feedback loops containing growth rates of labour productivity, employment ratio as well as employment ratio itself. Secondly, the respective loops' polarity is to be determined.

There are one negative feedback loop and one positive feedback loop for the rate of change of labour productivity:

loop Number 1 of length 3 – negative: $\hat{a}\xrightarrow{-}\dot{u}\rightarrow u\rightarrow\widehat{k/l}\rightarrow\hat{a}$;

loop Number 2 of length 11 – *positive*:

$$\hat{a}\xrightarrow{-}\hat{\sigma}\rightarrow\dot{\sigma}\rightarrow\sigma\xrightarrow{-}\frac{1-u}{\sigma}\rightarrow\hat{v}\rightarrow\dot{v}\rightarrow v\rightarrow\hat{w}\rightarrow\hat{u}\rightarrow\dot{u}\rightarrow u\rightarrow\widehat{k/l}\rightarrow\hat{a}.$$

There are three additional negative feedback loops for the rate of change of employment ratio:

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a widespread mistreatment of specification errors in econometrics in (Blatt 1983: 335–349).

loop Number 1 of length 6 – negative: $\hat{v} \to \dot{v} \to v \to \hat{w} \to \dot{u} \to u \xrightarrow{-} \frac{1-u}{\sigma} \to \hat{v}$;

loop Number 2 of length 6 – negative: $\hat{v} \to \dot{v} \to v \to \hat{w} \to \hat{u} \to u \to k\hat{/}l \xrightarrow{-} \hat{v}$;
loop Number 3 of length 10 – negative:
$\hat{v} \to \dot{v} \to v \to \hat{w} \to \dot{u} \to u \to k\hat{/}l \to \hat{\sigma} \to \dot{\sigma} \to \sigma \xrightarrow{-} \frac{1-u}{\sigma} \to \hat{v}$.

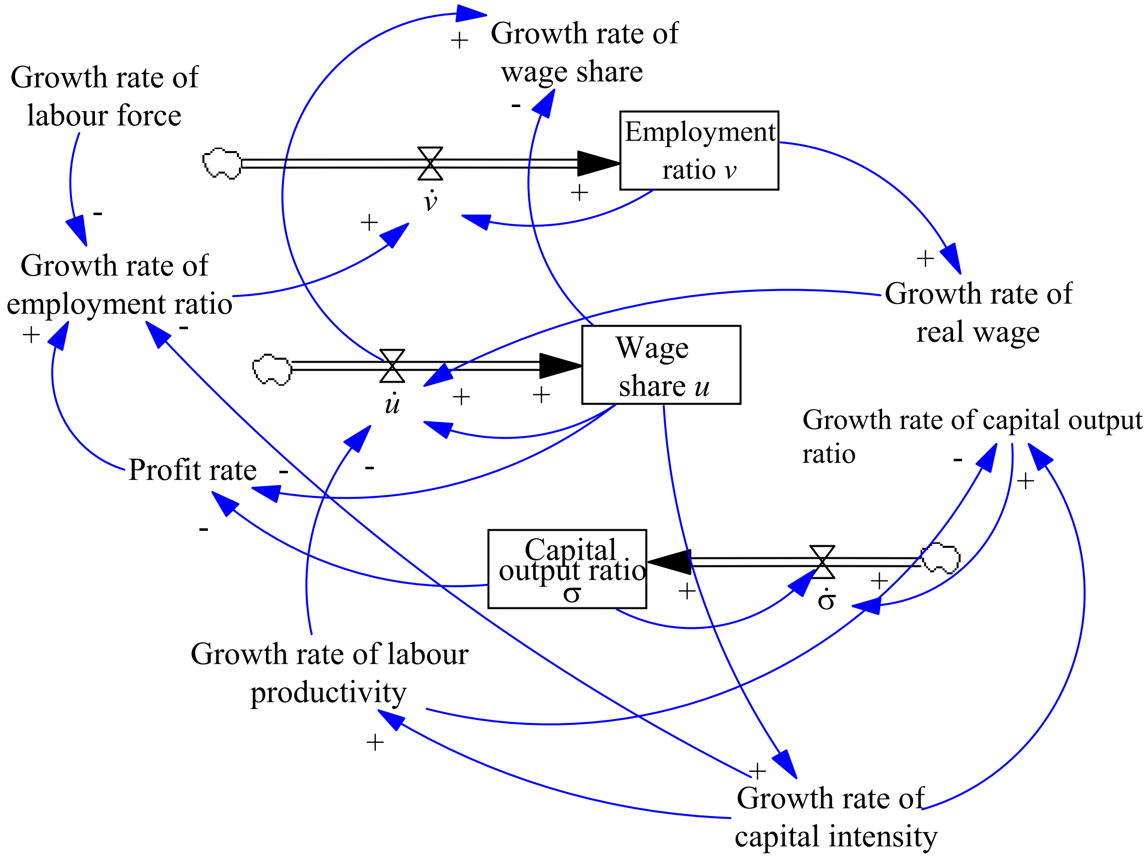

Figure 1b The causal-loop structure of KGM-II

All loops including employment ratio (v), except the loop presented on the Figure 5, are already included in the above loops.

The loop Number 2 of length 11 for v, $k\hat{/}l$ $\hat{a}$, $\hat{\sigma}$ and the other variables represents the only reinforcing roundabout increasing return. Therefore KGM-II includes reinforcing positive returns to scale *in embryo*, q.e.d.

Let us add a structural change – the direct positive scale effect – and trace out its consequences.

2 The hypothetical law (HL) of growth cycle

2.1 Developing the embryonic reinforcing increasing returns

The direct positive scale effect (a positive partial derivative of growth rate of labour productivity with respect to growth rate of employment ratio $\frac{\partial \hat{a}}{\partial \hat{v}} > 0$) manifests itself in a significant positive correlation between the respective random variables (annual growth rates of labour productivity and employment ratio): about 0.62 for the USA economy over 1970–2000. This empirical finding shows that the trade-off between employment and labour productivity, implied by KGM-I, is refuted in the long-term. Although this trade-off may apparently prevail at micro level in many instances, the economy of scale typically dominates at macro level.

The modified technical progress function is represented by the equation (5b) for $\hat{a}$ that includes additionally $\hat{v}$ in the simplest possible way

$$\hat{a} = \Omega(\widehat{k / l}) + m\hat{v}, \tag{5b}$$

where m > 0. This equation holds the sign of the partial derivative $\frac{\partial \hat{a}}{\partial \hat{\sigma}} > 0$ as in the former technical progress function (5a) of KGM-II; it also explains the observed spurious negative correlation between annual growth rates of capital-output ratio and labour productivity (Figure 7, Table 1) as the combined effects of *all* variables affecting growth rate of labour productivity.

The intensive deterministic form of HL for the system of economic growth, class struggle, induced technical change and increasing returns consists of the equations (3a), (9) and (10a)

$$\dot{\sigma} = -[\psi(u) + m\hat{v}]\sigma, \tag{3a}$$

$$\dot{v} = [x\frac{1-u}{\sigma} + \psi(u) - \phi(\psi(u)) - \beta]v, \tag{9}$$

$$\dot{u} = [-\gamma + \rho v - \phi(\psi(u)) - m\hat{v}]u. \tag{10a}$$

Adding $\hat{v}$ does not affect the stationary state: (11a) ≡ (11).

2.2 The causal-loop structure of HL

Figure 1c portrays the causal-loop structure of HL. Unlike KGM-I and KGM-II, it contains direct increasing return since the growth in employment ratio facilitates the growth in labour productivity.

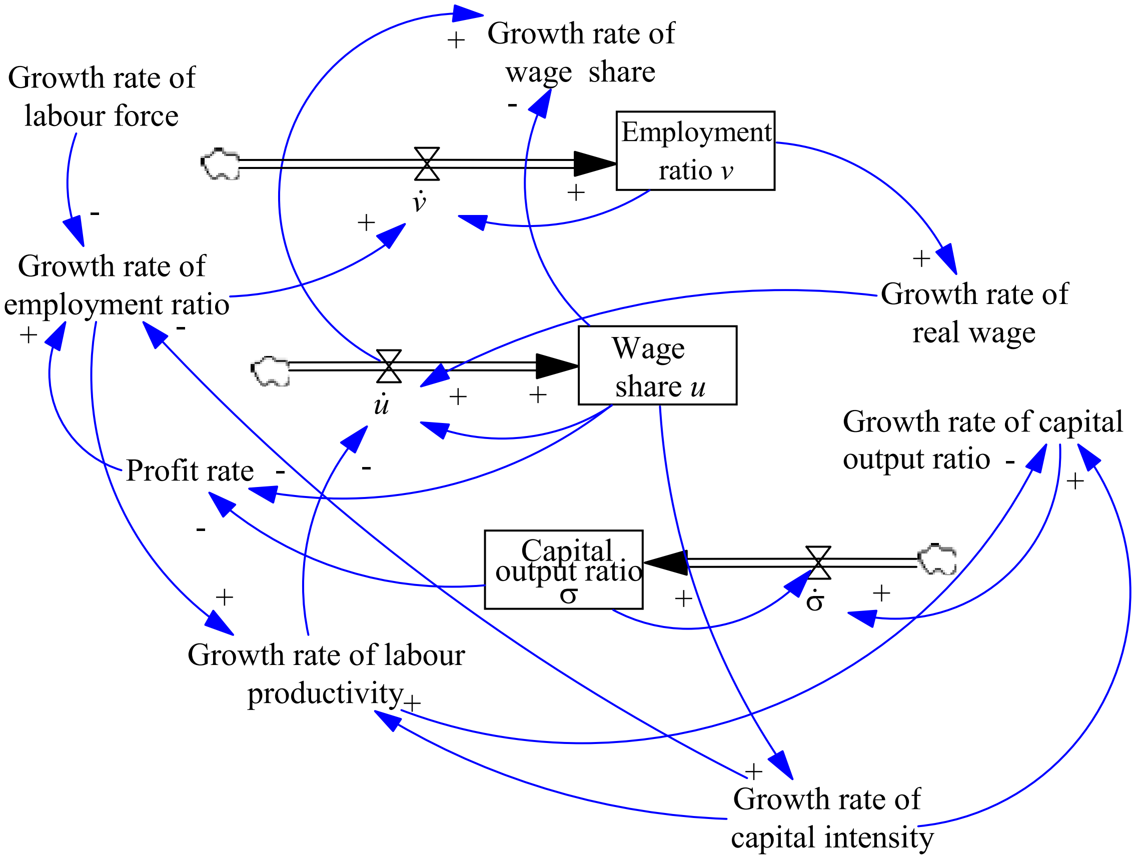

Figure 1c The causal-loop structure of HL

The above structural change generates additional positive and negative feedback loops for $\hat{v}$:

loop Number 1 of length 5 – positive: $\hat{v} \to \hat{a} \xrightarrow{-} \dot{u} \to u \to \widehat{k/l} \xrightarrow{-} v \to \hat{v}$;

loop Number 2 of length 5 – positive: $\hat{v} \to \hat{a} \xrightarrow{-} \dot{\sigma} \to \sigma \xrightarrow{-} \frac{1-u}{\sigma} \to \dot{v} \to \hat{v}$;

loop Number 3 of length 5 – positive: $\hat{v} \to \hat{a} \xrightarrow{-} \dot{u} \to u \xrightarrow{-} \frac{1-u}{\sigma} \to \dot{v} \to \hat{v}$;

loop Number 4 of length 6 – negative: $\hat{v} \to \hat{a} \xrightarrow{-} \dot{u} \to u \to \widehat{k/l} \xrightarrow{-} \dot{v} \to v \xrightarrow{-} \hat{v}$;

loop Number 5 of length 6 – negative: $\hat{v} \to \hat{a} \xrightarrow{-} \dot{\sigma} \to \sigma \xrightarrow{-} \frac{1-u}{\sigma} \to \dot{v} \to v \xrightarrow{-} \hat{v}$;

loop Number 6 of length 6 – negative: $\hat{v} \to \hat{a} \xrightarrow{-} \dot{u} \to u \xrightarrow{-} \frac{1-u}{\sigma} \to \dot{v} \to v \xrightarrow{-} \hat{v}$;

loop Number 7 of length 8 – positive:

$\hat{v} \to \hat{a} \xrightarrow{-} \dot{u} \to u \to \widehat{k/l} \to \dot{\sigma} \to \sigma \xrightarrow{-} \frac{1-u}{\sigma} \to \dot{v} \to \hat{v}$;

loop Number 8 of length 9 – negative:

$\hat{v} \to \hat{a} \xrightarrow{-} \dot{u} \to u \to \widehat{k/l} \to \dot{\sigma} \to \sigma \xrightarrow{-} \frac{1-u}{\sigma} \to \dot{v} \to v \xrightarrow{-} \hat{v}$.

The only positive loop Number 9 of length 11 of KGM-II remains in the newly modified model: $\hat{v} \to \dot{v} \to v \to \hat{w} \to \dot{u} \to u \to \widehat{k/l} \to \hat{a} \xrightarrow{-} \hat{\sigma} \to \dot{\sigma} \to \sigma \xrightarrow{-} \frac{1-u}{\sigma} \to \hat{v}$.

The system of the equations (3a), (9) and (10a) also represents a *competitive-co-operative* system in the three-dimensional phase space:

$$\partial f_1(\sigma)/\partial v = 0,\ \partial f_1(\sigma)/\partial u = [-\psi'(u) + m(-\frac{\psi'(u)}{u} + x\frac{1}{\sigma})]\sigma > 0;$$

$$\partial f_2(v)/\partial \sigma = -x\frac{1-u}{\sigma^2}v < 0,\ \partial f_2(v)/\partial u = \{\psi'(u) - \phi'[\psi(u)]\psi'(u) - x\frac{1}{\sigma}\}\, v < 0;$$

$$\partial f_3(u)/\partial \sigma = mx\frac{1-u}{\sigma^2}u > 0 \text{ (it is the new non-zero element)},\quad \partial f_3(u)/\partial v = \rho u > 0.$$

The Figure 8 reports on the additional (in fact, first main) positive feedback loop in this latter competitive – co-operative system. The substantial number of reinforcing scale effects uncovered may destabilize the stationary growth.

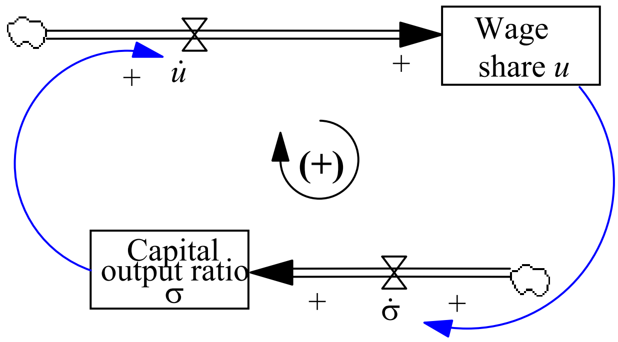

Figure 8 The first main positive loop of the intensive form of HL

When the latter effects are strong enough in HL, the dynamic stationary state is locally repelling and bifurcates into closed orbits. Their period is estimated by the equation (31) in the Appendix. The Andronov – Hopf bifurcation does take place in the system (3a), (9) and (10a) at some positive magnitude of the control parameter $m_0 > 0$ (the equation (26) in Section A1.1).

3 A Synthesis of the control law (CL)

In view of the present author, successful macroeconomic policy requires strengthening elements of feed-forward control over capital accumulation and primary income distribution. Feed-forward control, as known, changes variables according to expected future states of the economy.

Rate of profit is the well-known key instrument of business control that does not require explanation in this paper. Still an important aspect deserves attention. It is reasonable to add a new negative feedback loop (Figure 9), containing only one level variable, namely wage share (u), to the structure comprising HL. This additional loop provides stronger grip over the profit rate.

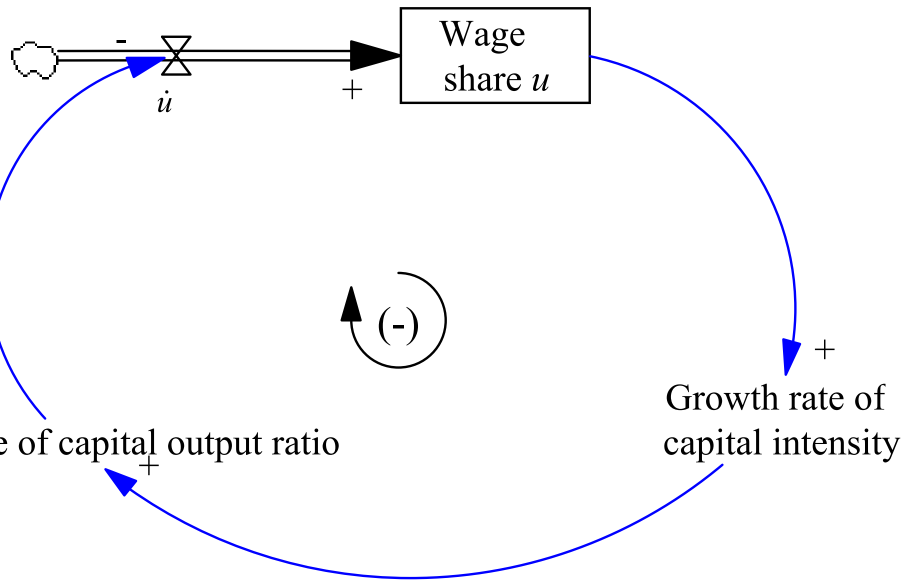

Figure 9 The new negative feedback loop controlling profitability via a single level variable (wage share)

The rate of change of wage share becomes negatively dependent on the rate of change of capital-output ratio, which is a leading macroeconomic indicator. In particular, the growing capital-output ratio and increasing unemployment in crises are to be compensated by a substantial reduction in unit labour cost (relative wage). On the other hand, the targeted growth rate of profitability has to decline when the real employment ratio approaches the desired employment ratio for avoiding over-extending of the controlled economy.

Assume that the decision-makers (the State officials, owners of capital, labourers) set a desirable growth rate of profitability depending on a difference between indicated and current employment ratios:

$$\Pi = \frac{\dot{r}}{r} = \frac{-\dot{u}}{1-u} - \hat{\sigma} = c_1 + c_2(X - v), \tag{17}$$

where $v < X$ is typical for recessions and depressions; it is assumed that $c_1 = 0$ for simplicity. For efficient closed loop control, the parameter c_2 has to be, of course, positive.

This employment and profitability targeting brings about the new patterns of primary income distribution whereby the equation (18) determines time derivative of wage share and the equation (19) – the rate of change of real wage:

$$\dot{u} = \left[-\hat{\sigma} - c_2(X - v)\right](1-u), \tag{18}$$

$$\hat{w} = \hat{u} + \hat{a} = \left[-\hat{\sigma} + c_2(v - X)\right]\frac{1-u}{u} + \hat{a} = [-\hat{k/l} + c_2(v - X)]\frac{1-u}{u} + \frac{\hat{a}}{u}. \tag{19}$$

On the one hand, a rise in the employment ratio and an increase in the growth rate of labour productivity facilitate the growth rate of real wage. The effect of labour productivity on real wage is in agreement with "ability–to–pay" hypothesis: firms are more likely to grant wage increases when there are productivity increases.

On the other hand, an increase in the growth rate of capital intensity impedes the growth rate of real wage. Notice that the rate of surplus value ($\frac{1-u}{u}$) and the indicated employment ratio (X) affect the growth rate of the real wage in the equation (19) as well. Due to this combined control the dependence of workers' real wage on productivity is not destabilising contrasted to an earlier model of never-ending growth cycles based on KGM-I (Ploeg 1987).

The equivalent form of the equation (18) transformed by usage of the equation (3a) is the equation (10c). The intensive deterministic form of synthesised control law (CL) is the system (3a), (9) and (10c)

$$\dot{u} = [\psi(u) + m\hat{v} + c_2(v - X)](1-u) . \tag{10c}$$

At a fixed-point ($\dot{\sigma}$, $\dot{v}$, $\dot{u}$) = 0.

The Harrod – Domar condition gives the non-trivial stationary state of this system $E_c = (\sigma_c, v_c, u_c)$:

$$\sigma_c \equiv \sigma_a = x(1-u_a)/(\alpha_a + \beta),\ v_c = X,$$

$$u_c \equiv u_a = \psi^{-1}(0) = \frac{1}{1-\phi'(0)},^{6} \tag{11c}$$

where the stationary growth rate of labour productivity is $\alpha_c \equiv \alpha_a = \phi(0)$. Note that E_c is different from E_a if $v_c = X \neq v_a$; a goal of the pro-growth stabilisation policy could be attaining

$1 > v_c = X > v_a$.

The Figure 1d presents the causal-loop structure of CL. Like KGM-II and HL, CL includes the growth rate of capital intensity. The new most important elements are the indicated (desired) employment ratio (X) and targeted growth rate of profitability (Π).

The system (3a), (9), (10c) represents a *competitive-co-operative* system in a three-dimensional phase space:

$$\partial f_1(\sigma)/\partial v = 0,\ \partial f_1(\sigma)/\partial u = -\psi'(u)\sigma > 0;$$

$$\partial f_2(v)/\partial \sigma = -x\frac{1-u}{\sigma^2}v < 0,\ \partial f_2(v)/\partial u = \{\psi'(u) - \phi'[\psi(u)]\psi'(u) - x\frac{1}{\sigma}\}v < 0;$$

$$\partial f_3(u)/\partial \sigma = -mx\frac{(1-u)^2}{\sigma^2} < 0,\quad \partial f_3(u)/\partial v = c_2(1-u) > 0.$$

The all three partial derivatives of wage share differ from the respective ones in the previous models. In particular, the negative $\frac{\partial \dot{u}}{\partial \sigma}$ substitutes in CL the positive $\frac{\partial \dot{u}}{\partial \sigma}$ in HL. This competitive – co-operative system includes the additional negative feedback loop (Figure 10) and substantially modified third loop with alternating polarity (Figure 11). The substitution of $\frac{\partial \dot{u}}{\partial v} = \rho u$ in KGM-I, KGM-II and CL by $\frac{\partial \dot{u}}{\partial v} = c_2(1-u) > 0$ in CL is also worth of noticing as it effects the feedback loops gains.

[6] The derivative holds the initial property $\psi'(u_c) = \frac{[1-\phi'(0)]^2}{\phi''} < 0$ as assumed in KGM-I.

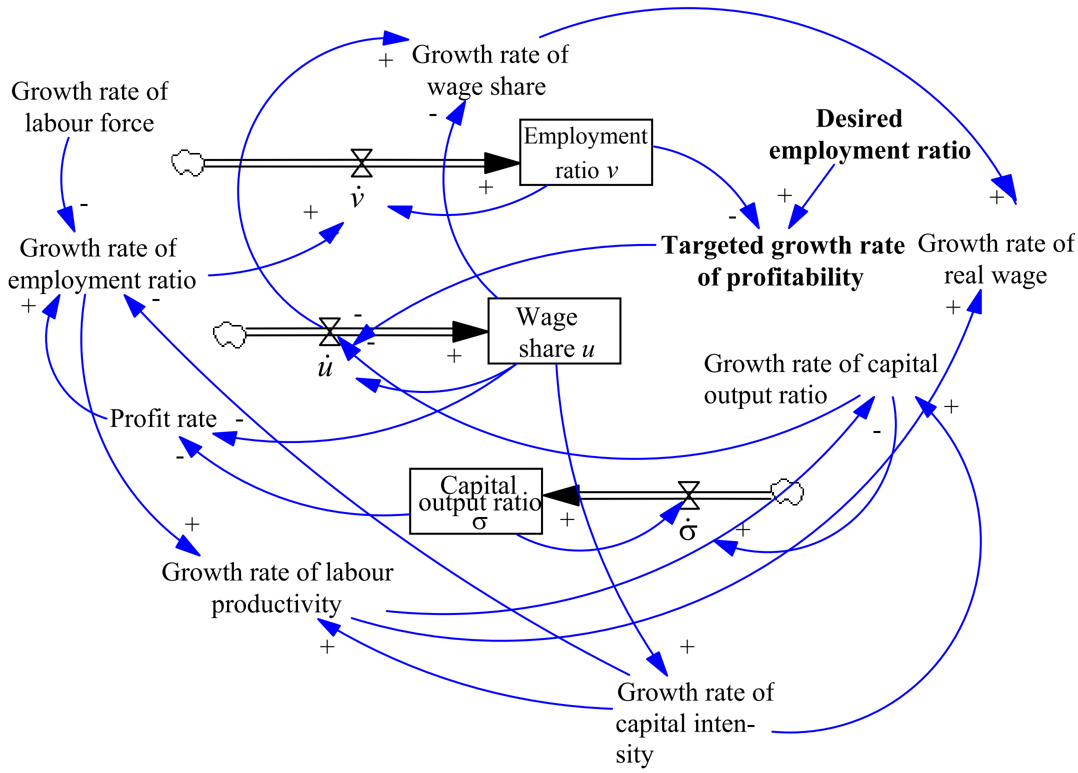

Figure 1d The causal-loop structure of CL

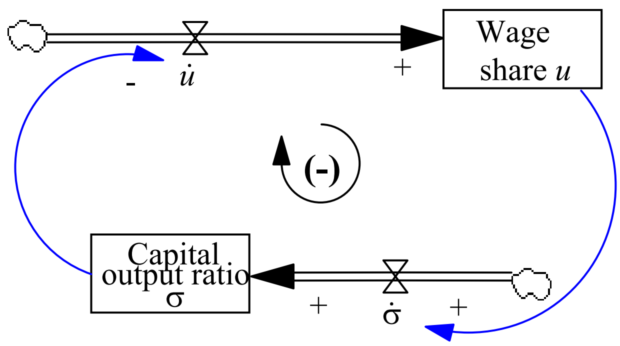

Figure 10 The additional negative loop of CL

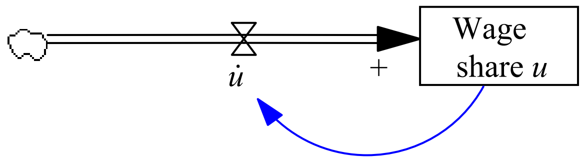

Figure 11 The modified third loop with alternating polarity:

$$\operatorname{sign}\{[\psi'(u)+m(\frac{\psi'(u)}{u}-x\frac{1}{\sigma})](1-u)-\frac{\dot{u}}{1-u}\}$$

The CL contains a greater number of negative feedback loops connecting growth rates of employment ratio and labour productivity than HL. Such negative feedback loops weaken the increasing returns. Still there are reinforcing increasing returns as well that enter into the two positive feedback loops among the following ones:

loop Number 1 of length 5 – *positive*: $\hat{v} \to \hat{a} \xrightarrow{-} \hat{\sigma} \to \dot{\sigma} \to \sigma \xrightarrow{-} \dfrac{1-u}{\sigma} \to \hat{v}$;

loop Number 2 of length 5 – negative: $\hat{v} \to \hat{a} \xrightarrow{-} \hat{\sigma} \xrightarrow{-} \dot{u} \to u \xrightarrow{-} \dfrac{1-u}{\sigma} \to \hat{v}$;

loop Number 3 of length 5 – negative: $\hat{v} \to \hat{a} \xrightarrow{-} \hat{\sigma} \xrightarrow{-} \dot{u} \to u \to \widehat{k/l} \xrightarrow{-} \hat{v}$;

loop Number 4 of length 6 – negative: $\hat{v} \to \dot{v} \to v \xrightarrow{-} \Pi \xrightarrow{-} \dot{u} \to u \xrightarrow{-} \dfrac{1-u}{\sigma} \to \hat{v}$;

loop Number 5 of length 6 – negative: $\hat{v} \to \dot{v} \to v \xrightarrow{-} \Pi \xrightarrow{-} \dot{u} \to u \to \widehat{k/l} \xrightarrow{-} \hat{v}$;

loop Number 6 of length 10 – negative:

$$\hat{v} \to \dot{v} \to v \xrightarrow{-} \Pi \xrightarrow{-} \dot{u} \to u \to \widehat{k/l} \to \dot{\sigma} \to \sigma \xrightarrow{-} \frac{1-u}{\sigma} \to \hat{v};$$

loop Number 7 of length 11 – *positive*:

$$\hat{v} \to \dot{v} \to v \xrightarrow{-} \Pi \xrightarrow{-} \dot{u} \to u \to \widehat{k/l} \to \hat{a} \xrightarrow{-} \hat{\sigma} \to \dot{\sigma} \to \sigma \xrightarrow{-} \frac{1-u}{\sigma} \to \hat{v}.$$

The latter loop is similar to the abovementioned positive loop already present in KGM-II and in HL. In spite of this similarity these two loops differ from each other because the link of the employment ratio and time derivative of wage share is intermediated by the targeted growth rate of profitability in CL (Figure 1d) unlike KGM-I, KGM-II and HL that do not have this variable. The loop No. 7 is accompanied by the loop No. 1 that is similar to the positive loop No. 2 in HL.

For strong enough direct scale effect (i.e., for m substantially higher than the bifurcation magnitude $m_0 > 0$ in HL), CL implies positive correlation between the growth rates of employment ratio and labour productivity as well as spurious negative correlation between growth rates of capital-output ratio and labour productivity. This reader finds above the empirical support for these theoretical implications (Sections 2.1 and 1.5).

All loops including employment ratio (v), except the loop presented on the Figure 5, are included in the above loops. There are direct and reinforcing roundabout increasing returns in HL, unlike KGM-I. Now, due to employment and profitability targeting, the strong direct scale effect favours pro-growth stabilization policy (cf. Ryzhenkov 2000: 103; 2006: 36–38).

The proposed closed-loop control would be destabilizing if the direct scale effect were substantially negative. Then the dynamic stationary state is locally repelling and bifurcates into closed orbits. Their period is estimated by the equation (38) in the Appendix. The Andronov – Hopf bifurcation does take place in the system (3a), (9) and (10c) at some negative magnitude of the control parameter $m_0 < 0$ (the equation (37) in Section A2.2).

Conclusion

Economic systems are highly complex control systems. Complexity results mainly from high interrelatedness of basic elements, non-linearity and delays (Milling 2001). A feedback loop constitutes the central building block in these complex systems.

This paper touches upon some unsettled essential issues, especially concerning the notion of increasing return (economy of scale), that have an indisputable practical importance. In this kind of problem solving a formalized decision support can successfully be applied. To emphasize the top executive perspective of macroeconomic control, the term Policy Support System (PSS) is pertinent (Milling 1990). PSS is concerned with the provision of computer assistance beyond the classical areas of programmed decision-making. PSS serves to understand better objectives and to improve decision-making.

This paper demonstrates that the Kennedy – Goodwin macroeconomic model of growth cycles with induced technical change contains only weakening roundabout increasing returns that explains the local asymptotical stability of the stationary state. This model is changed to allow for direct increasing return to scale, whereby the growth rate of employment ratio positively influences the growth rate of labour productivity. If the latter effect is strong enough, the dynamic stationary state is locally repelling and bifurcates into closed orbits. Their period is estimated.

This paper supposes a closed loop control that stabilizes the oscillatory dynamics of the main macroeconomic variables, sustaining total profit and employment under direct and reinforcing roundabout increasing returns. It is proved that the supposed policy would be destabilizing if the direct scale effect were powerfully negative that is not correct empirically.

The application and further development of the control law (CL) maintained by PSS is helpful for conceiving a more efficient policy of primarily income distribution that stabilizes the oscillatory dynamics of the main macroeconomic variables, maintaining profitability and employment.

The global analysis of the above models requires substantial additional efforts beyond the scope of this paper. Further research will deepen understanding of capital accumulation under increasing returns.

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APPENDIX

A1 The local analysis of the hypothetic law (HL)

A1.1 Asymptotic stability of the stationary state

PROPOSITION 1.
The dynamics of the system (3a), (9), (10a) in the neighbourhood of its stationary state (11a) are Poincaré (locally) stable provided that the inequality (C-3m) holds and is unstable otherwise.
Proof.
The Jacoby matrix evaluated at the stationary state (11a) for the system (3a), (9) and (10a) in a simpler form is

$J_a =$

$$\begin{array}{|c|c|c|} \hline md & 0 & [-\psi'(u_a) + m(-\frac{\psi'(u_a)}{u_a} + x\frac{1}{\sigma_a})]\sigma_a \\ \hline -\frac{d}{\sigma_a} v_a & 0 & (\frac{\psi'(u_a)}{u_a} - x\frac{1}{\sigma_a})v_a \\ \hline m\frac{d}{\sigma_a}u_a & \rho u_a & [-\phi'(0)\psi'(u_a) + m(-\frac{\psi'(u_a)}{u_a} + x\frac{1}{\sigma_a})]u_a \\ \hline \end{array}, \quad (15a)$$

where the stationary growth rate of net output $d = x\frac{1-u_a}{\sigma_a} > 0$, $\phi'(0) = \frac{u_a - 1}{u_a} < 0$.

The characteristic polynomial equals now

$$\lambda^3 + a_2(m)\lambda^2 + a_1(m)\lambda + a_0(m) = 0, \quad (16\text{-m})$$

where the coefficients are functions of the parameter m.

The Routh – Hourwitz necessary and sufficient conditions for the local stability are

$a_0(m) > 0$, (C-1m)

$a_1(m) > 0$, (C-2m)

$a_1(m)a_2(m) > a_0(m)$, (C-3m)

similar to the conditions (C-1), (C-2) and (C-3) in the case of KGM-I (Section 1.3). These requirements confine the region $S \subset R$ such that for $m \in S$ the stationary state E_a of the system (3a), (9) and (10a) is locally asymptotically stable.

Find the explicit expressions for the parameters of the characteristic polynomial (16-m):

$$a_0(m) = -J_{32}(J_{21}J_{13} - J_{23}J_{11})$$

$$= -\rho u_a \{ -\frac{d}{\sigma_a} v_a [-\psi'(u_a) + m(\frac{-\psi'(u_a)}{u_a} + x\frac{1}{\sigma_a})]\sigma_a - [\psi'(u_a)\frac{1}{u_a} - x\frac{1}{\sigma_a}]v_a md \}$$

$$= -\rho u_a \{ -dv_a [-\psi'(u_a) + m(-\frac{\psi'(u_a)}{u_a} + x\frac{1}{\sigma_a})] + (-\frac{\psi'(u_a)}{u_a} + x\frac{1}{\sigma_a})v_a md \}$$

$$= -\rho u_a(-dv_a)[-\psi'(u_a)] = -\rho u_a\, dv_a\, \psi'(u_a) > 0. \quad (20)$$

This inequality does not depend on m and x: $\frac{\partial a_0}{\partial m} = 0$, $\frac{\partial a_0}{\partial x} = 0$. So the condition (C-1m) is satisfied.

Consider the function

$$a_1(m) = -(J_{23}J_{32} + J_{13}J_{31} - J_{33}J_{11})$$

$$= -\{ (\frac{\psi'(u_a)}{u_a} - x\frac{1}{\sigma_a})v_a\, \rho u_a + [-\psi'(u_a) + m(-\frac{\psi'(u_a)}{u_a} + x\frac{1}{\sigma_a})]\sigma_a\, m\frac{d}{\sigma_a}u_a$$

$$-[-\phi'(0)\psi'(u_a) + m(-\frac{\psi'(u_a)}{u_a} + x\frac{1}{\sigma_a})]u_a\, md\}$$

$$= -\{ (\frac{\psi'(u_a)}{u_a} - x\frac{1}{\sigma_a})v_a\, \rho u_a + [-\psi'(u_a) + m(-\frac{\psi'(u_a)}{u_a} + x\frac{1}{\sigma_a})]\, mdu_a$$

$$-[-\phi'(0)\psi'(u_a) + m(-\frac{\psi'(u_a)}{u_a} + x\frac{1}{\sigma_a})]u_a\, md \}$$

$$=-\{(\frac{\psi'(u_a)}{u_a}-x\frac{1}{\sigma_a})v_a\,\rho u_a+[-\psi'(u_a)]\,mdu_a-[-\phi'(0)\psi'(u_a)]u_a\,md\}$$

$$=-\{(\frac{\psi'(u_a)}{u_a}-x\frac{1}{\sigma_a})v_a\,\rho u_a+[-\psi'(u_a)+\frac{u_a-1}{u_a}\psi'(u_a)]u_a\,md\}$$

$$=-\{(\frac{\psi'(u_a)}{u_a}-x\frac{1}{\sigma_a})v_a\,\rho u_a+[-\psi'(u_a)\,(1-\frac{u_a-1}{u_a}]u_a\,md\}$$

$$=-\{(\frac{\psi'(u_a)}{u_a}-x\frac{1}{\sigma_a})v_a\,\rho u_a-\psi'(u_a)\,md\}. \qquad (21)$$

It is decreasing linear function of m independent of x:

$\frac{\partial a_1}{\partial m}=\psi'(u_a)d<0$; in particular, $a_1(m)>0$ for $m\le 0$ and $a_1(m)=0$ for

$$m_1=\frac{\left[\frac{\psi'(u_a)}{u_a}-x\frac{1}{\sigma_a}\right]\rho v_a u_a}{\psi'(u_a)d}=\frac{\left[\frac{\psi'(u_a)}{u_a}-x\frac{1}{\sigma_a}\right]\rho v_a u_a}{\psi'(u_a)x\frac{1-u_a}{\sigma_a}}$$

$$=\frac{[\psi'(u_a)\sigma_a-xu_a]\rho v_a}{\psi'(u_a)x(1-u_a)}=\frac{\left[\sigma_a-\frac{xu_a}{\psi'(u_a)}\right]\rho v_a}{x(1-u_a)}>0. \qquad (22)$$

We see the condition (C-2m) is valid for $m\in(-\infty,m_1)$, where m_1 is determined by (22). Before establishing explicit requirements of the remaining condition (C-3m), we investigate properties of the function

$$a_2(m)=-(J_{11}+J_{33})=-\{md+[-\phi'(0)\psi'(u_a)+m(-\frac{\psi'(u_a)}{u_a}+x\frac{1}{\sigma_a})]u_a\}$$

$$=-\{md+[-\phi'(0)\psi'(u_a)+m(-\frac{\psi'(u_a)}{u_a}+x\frac{1}{\sigma_a})]u_a\}$$

$$=-\{md+(1-u_a)\psi'(u_a)+[m(-\frac{\psi'(u_a)}{u_a}+x\frac{1}{\sigma_a})]u_a\}. \qquad (23)$$

It is a decreasing linear function of m:

$\frac{\partial a_2}{\partial m}<0$; in particular, $a_2(m)>0$ for $m\le 0$, $a_2(m_2)=0$, where

$$m_2=-\frac{\psi'(u_a)(1-u_a)}{d-\left[\frac{\psi'(u_a)}{u_a}-x\frac{1}{\sigma_a}\right]u_a}=-\frac{\psi'(u_a)(1-u_a)}{x\frac{1-u_a}{\sigma_a}-\left[\psi'(u_a)-x\frac{u_a}{\sigma_a}\right]}$$

$$=-\frac{\psi'(u_a)(1-u_a)}{x\frac{1}{\sigma_a}-\psi'(u_a)}>0, \qquad (24)$$

and $a_2(m)<0$ for $m>m_2$.

The necessary requirement for (C-3m) being true is $m\in(-\infty,m_2)$, where m_2 is determined by (24). This requirement is not sufficient. Indeed, take a closer look at the equation (25) that defines the function

$$a(m) = a_1a_2 - a_0 = (J_{23}J_{32} + J_{13}J_{31} - J_{33}J_{11})(J_{11} + J_{33}) + J_{32}(J_{21}J_{13} - J_{23}J_{11})$$

$$= \{(\frac{\psi'(u_a)}{u_a} - x\frac{1}{\sigma_a})v_a\, \rho u_a - \psi'(u_a)md\}\{md + [-\phi'(0)\psi'(u_a) + m(-\frac{\psi'(u_a)}{u_a} + x\frac{1}{\sigma_a})]u_a\}$$

$$+\ \rho u_a\, dv_a\, \psi'(u_a)$$

$$= [(\psi'(u_a) - x\frac{u_a}{\sigma_a})v_a\rho - \psi'(u_a)md]\{md + [(1-u_a)\psi'(u_a) + m(-\psi'(u_a) + x\frac{u_a}{\sigma_a})]\}$$

$$+\rho u_a\, dv_a\, \psi'(u_a)\,. \tag{25}$$

Lemma 1. The function $a(m) > 0$ for for $m \in (-\infty, 0]$.

Proof. Both $a_1(m)$ and $a_2(m)$ are positive and declining linear functions of m. Keep in mind that $\frac{\partial a_0}{\partial m} = 0$ (for any m); $\frac{\partial a_1}{\partial m} < 0$, $a_1(m) > 0$ and $\frac{\partial a_2}{\partial m} < 0$, $a_2(m) > 0$ for $m \le 0$, so $\frac{\partial a(m)}{\partial m} = \frac{\partial a_1(m)}{\partial m}a_2(m) + a_1(m)\frac{\partial a_2(m)}{\partial m} - \frac{\partial a_0}{\partial m} = \frac{\partial a_1(m)}{\partial m}a_2(m) + a_1(m)\frac{\partial a_2(m)}{\partial m} - 0 < 0$ (for $m \le 0$). Moreover, $a(0) > 0$ as in KGM-I. Therefore $a(m) > 0$ for $m \in (-\infty, 0]$.[7]

Lemma 2. The conjugate roots of the quadratic equation $a(m) = 0$ are $m_0 \in (0, \min(m_1, m_2))$ and $m_3 > \max(m_1, m_2) > 0$. Therefore the four critical magnitudes of this control parameter are such that $0 < m_0 < \min(m_1, m_2) < \max(m_1, m_2) < m_3$.

Proof.

Recall the continuous function $a(m)$ is a polynomial of the second degree. Consider this function on a segment $[0, \max(m_1, m_2)]$.

I. Let $m_1 \le m_2$. Consider the function $a(m)$ on the segment $[0, m_1]$. The function $a(m)$ has magnitudes of opposite sign at the ends of this segment: $a(0) > 0$ as proved for KGM-I (Shah, Desai 1981); $a(m_1) = -a_0 < 0$ as $a_1(m_1) = 0$. Therefore according to the well-known theorem on a function $\varphi(c)$ that is continuous on a segment $[a, b]$ and $\varphi(a)\varphi(b) < 0$, $\exists$ at least one point c ($a < c < b$) such, that $\varphi(c) = 0$. So there exists at least one m ($0 < m_0 < m_1$) such that $a(m_0) = 0$. As the function $a(m)$ is declining monotonically on the segment $[0, m_1]$ there is no other solution. So there $\exists$ only one solution m_0 ($0 < m_0 < m_1$) such that $a(m_0) = 0$.

II. Let $m_1 \le m_2$. Consider the function $a(m)$ on the segment $[0, m_2]$. The function $a(m)$ has magnitudes of opposite sign at the ends of this segment: $a(0) > 0$; $a(m_2) = -a_0 < 0$ as $a_2(m_2) = 0$. Therefore according to the same theorem on a function $\varphi(c)$ that is continuous on a segment $[a, b]$ and $\varphi(a)\varphi(b) < 0$, $\exists$ at least one point c ($a < c < b$) such, that $\varphi(c) = 0$. So there exists at least one m ($0 < m < m_2$) such that $a(m) = 0$. As this function is declining monotonically on the segment $[0, m_2]$ there could be no more solutions. So $\exists$ only one solution m_* ($0 < m_* < m_2$) such that $a(m_*) = 0$.

It follows from I and II that $m_0 = m_*$.

III. Let $m_1 > m_2$. Than the same considerations lead to the conclusion that the only solution of $a(m) = 0$ is $m_0 \in (0, \min(m_1, m_2))$.

Thus the only solution for $a(m) = 0$ is m_0 such that $0 < m_0 < \min(m_1, m_2)$. The conjugate root of the quadratic equation $a(m) = 0$ is either $m_3 = m_0$ or $m_3 > \max(m_1, m_2) > 0$. In the latter

[7] Thus the Andronov – Hopf bifurcation is impossible for non-positive m.

case, $a_1(m_3) < 0$ and $a_2(m_3) < 0$, therefore the stationary state is not stable, but the Andronov – Hopf bifurcation theorem is not applicable for this case.

Now we will prove that the case $m_3 = m_0$ is excluded. Let

$a_1(m) = e - bm,$

$a_2(m) = c - hm,$

then

$a_1(m)a_2(m) - a_0 = (e - bm)(c - hm) - a_0 = 0,$

where $b = -\dfrac{\partial a_1}{\partial m} > 0$, $h = -\dfrac{\partial a_2}{\partial m} > 0$, $a_0 > 0$,

or

$bhm^2 - (bc + eh)m + ce - a_0 = 0.$

This quadratic equation has the solution(s)

$$m_{0,3} = \frac{bc + eh \pm \sqrt{(bc + eh)^2 - 4bh(ce - a_0)}}{2bh} = \frac{bc + eh \pm \sqrt{(bc - eh)^2 + 4bha_0}}{2bh}. \qquad (26)$$

For $m_0 = m_3$ both the non-negative terms inside the square root must equal zero $(bc - de)^2 = 0, \quad 4bda_0 = 0$. The latter is not possible as $b > 0$, $h > 0$, $a_0 > 0$. This excludes $m_3 = m_0$. Finally, $0 < m_0 < \min(m_1, m_2)$, $m_3 > \max(m_1, m_2) > 0$; $a(m) > 0$ for $m \in [0, m_0)$, where m_0 is determined by (26). Thus the stationary state E_a is stable for $m < m_0$. The inequality (C-3m) turns into equality $a(m) = 0$ when $m = m_0$, then E_a is not stable. The proof of the Proposition 1 is finished.[8]

A1.2 The Andronov – Hopf bifurcation in HL

The Hopf theorem is a tool for establishing the existence of closed orbits. In this study of the cyclical motion I choose m (see the equation (5b)) as a bifurcation (control) parameter.

Consider the stationary state of the system (3a), (9) and (10a) as dependent on the control parameter m:

$$\dot{x} = 0 = f(x, m). \qquad (27)$$

The determinant of the Jacoby matrix (J_a) differs from zero in our case for any possible equilibrium (x, m) as $a_0 > 0$ (independently of m). The implicit function theorem ensures that for every m in a neighbourhood $Br(m_0) \in R$ of the parameter value m_0 there exists a unique equilibrium x_a. Changes of m do not affect s_a, v_a and u_a.

We assume the following properties are satisfied:

(a) the components of the function $f(x, m)$, corresponding to the system (3a), (9) and (10a), are analytic (i.e. given by power series);

(b) the Jacobian J_a (14) has a pair of pure imaginary eigenvalues and no other eigenvalues with zero real parts (in this case $\lambda_1 = -a_2(m_0) = < 0$);

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[8] Notice the above critical magnitudes of the bifurcation parameter m does not depend on the capitalists investment ratio x ($0 < x \leq 1$) as a_1, a_2 and a_0 are independent of this ratio. Still this ratio is a factor of the stationary capital-output ratio s_a.

(c) the derivative $\frac{d(\mathrm{Re}\lambda_{2,3}(m))}{dm} > 0$ for $m = m_0$ (it is the transversality condition);

(d) the stationary state E_a is asymptotically stable (for $m < m_0$).

Then, according to the Hopf theorem, there exists some periodic solution bifurcating from $x_a(m_0)$ at $m = m_0$ and the period of fluctuations is about $2\pi/\beta_0$ ($\beta_0 = \lambda_{2,3}(m_0)/i$). If a closed orbit is an attractor, it is usually called a *limit cycle*.

The characteristic polynomial for $m = m_0$ is

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_1a_2 = \lambda^2(\lambda + a_2) + a_1(\lambda + a_2) =$$
$$(\lambda + a_2)(\lambda^2 + a_1) = 0. \tag{28}$$

It has the following roots:

$$\lambda_1 = -a_2 = < 0 \,; \tag{29}$$
$$\lambda_{2,3} = \pm i\sqrt{a_1} \,. \tag{30}$$

What remains is proving the transversality condition. The general formula $\frac{\partial(\mathrm{Re}\lambda_{2,3}(m))}{\partial m} = \frac{-\left[\frac{\partial a_1}{\partial m}a_2 + a_1\frac{\partial a_2}{\partial m}\right] + \frac{\partial a_0}{\partial m}}{2(a_1 + a_2^2)}$ has been derived in the paper (Ryzhenkov 2004: 170–172). Recall that both $a_1(m_0) > 0$ and $a_2(m_0) > 0$; at $m = m_0$ it is true that $\frac{\partial a_0}{\partial m} = 0$, $\frac{\partial a_1}{\partial m} = \psi'(u_a)d < 0$, $\frac{\partial a_2}{\partial m} = -\{d + (-\frac{\psi'(u_a)}{u_a} + \frac{1}{\sigma_a})u_a\} < 0$. Thus we have the ratio that is also positive at $m = m_0$: $\frac{\partial(\mathrm{Re}\lambda_{2,3}(m))}{\partial m} > 0$, q.e.d.[9]

The period of oscillations near E_a is about $2\pi/\sqrt{a_1(m_0)}$ (years). It depends on $m = m_0$.[10] The period of the cycle is approximately

$$T \approx \frac{2\pi}{\sqrt{[x\frac{u_a}{\sigma_a} - \psi'(u_a)]\rho v_a}} \tag{31}$$

as

$$a_1(m_0) = -\{(\frac{\psi'(u_a)}{u_a} - x\frac{1}{\sigma_a})v_a\,\rho u_a - \psi'(u_a)m_0\,d\} \approx -\{(\frac{\psi'(u_a)}{u_a} - x\frac{1}{\sigma_a})v_a\,\rho u_a$$

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[9] The Andronov – Hopf bifurcation does not take place at m_3, because E_a is not stable for $a_1(m_3) < 0$ and $a_2(m_3) < 0$.

[10] In the original Goodwin model all profits are invested ($x = 1$) and the capital-output ratio is an exogenously given constant ($\sigma = const > 1$). So a period of fluctuations near the stationary state (v_e, u_e) is about $2\pi/(\sigma^{-1}\rho v_e u_e)^{1/2}$ years, where $u_e = 1 - \sigma(\alpha + \beta)$, $v_e = (\alpha + \gamma)/\rho$. In a more general version of this model the positive investment ratio does not exceeds 1 ($0 < x \leq 1$). Then $u_e = 1 - \sigma(\alpha + \beta)/x$, $v_e = (\alpha + \gamma)/\rho$, a period is about $2\pi/(\sigma^{-1}\rho x v_e u_e)^{1/2}$ years, respectively.

$= -\{(\psi'(u_a) - \frac{du_a}{1-u_a})v_a\,\rho$, since $-\psi'(u_a)m_0 d$ is close to zero. This approximation does not depend on m.

The period of the growth cycle near the stationary state is the shorter, the higher are the employment ratio and workers bargaining power, relative wage, the steeper is the slope of the mechanization (automation function) with respect to the relative wage; this period is the longer, the higher is the capital-output ratio. It is independent of the capitalists investment ratio x ($0 < x \leq 1$).

We have proved that the Andronov – Hopf bifurcation does take place in the system (3a), (9) and (10a) at $m = m_0$. The Hopf theorem establishes only the existence of closed orbits in a neighbourhood of x_a at m_0, still it does not clarify the stability of orbits, which may arise on either side of m_0.[11] Without detailed knowledge of the specific functional form of the technical progress function, it cannot be determined whether a supercritical or sub-critical bifurcation occurs. A further increase of the parameter m is destabilizing in this model. If the direct positive returns to scale are strong enough, the economy experiences escalating class conflict. Next section demonstrates that this in not inevitable.

A2 The local analysis of the control law (CL)

A2.1 Asymptotic stability of the stationary state

PROPOSITION 2. The dynamics of CL – the system (3a), (9), (10c) – in the neighbourhood of its stationary state (11a) are Poincaré (locally) stable provided that the inequality (36) holds and is unstable otherwise.

Proof. The Jacoby matrix for the stationary state for the system (3a), (9) and (10c) is:

$J_c =$

md	0	$[-\psi'(u_a) - m(\frac{\psi'(u_a)}{u_a} - x\frac{1}{\sigma_a})]\sigma_a$
$-\frac{d}{\sigma_a}X$	0	$(\frac{\psi'(u_a)}{u_a} - x\frac{1}{\sigma_a})X$
$-mx\frac{(1-u_a)^2}{\sigma_a^{\,2}}$	$c_2(1-u_a)$	$[\psi'(u_a) + m(\frac{\psi'(u_a)}{u_a} - x\frac{1}{\sigma_a})](1-u_a)$

. (32)

Now the parameters of the characteristic polynomial (16-m) are

$$a_0 = -J_{32}(J_{21}J_{13} - J_{23}J_{11})$$

$$= -c_2(1-u_a)\,\{-\frac{d}{\sigma_a}X\,[-\psi'(u_a) - m(\frac{\psi'(u_a)}{u_a} - x\frac{1}{\sigma_a})]\sigma_a - (\frac{\psi'(u_a)}{u_a} - x\frac{1}{\sigma_a})X\,md\}$$

$$= -\,c_2(1-u_a)\;dX\;\psi'(u_a) = const > 0, \tag{33}$$

it does not depend on m and x as in HL;

$$a_1 = -(J_{23}J_{32} + J_{13}J_{31} - J_{33}J_{11}) = \;-J_{23}J_{32}$$

[11] A particular closed orbit as well as frequency and amplitude of fluctuations along this closed orbit may depend on an initial point x_0. A closed orbit may even cross the upper economic limits of v and u in the phase space in this model.

$$=-\{(\frac{\psi'(u_a)}{u_a}-x\frac{1}{\sigma_a})X\ c_2(1-u_a)+[\psi'(u_a)+m(\frac{\psi'(u_a)}{u_a}-x\frac{1}{\sigma_a})]\sigma_a\, md^2$$
$$-[\psi'(u_a)+m(\frac{\psi'(u_a)}{u_a}-x\frac{1}{\sigma_a})](1-u_a)\,md\}$$
$$=-(\frac{\psi'(u_a)}{u_a}-x\frac{1}{\sigma_a})\ c_2(1-u_a)X\ >0,^{12} \tag{34}$$

it does not depend on x as in HL, still it does not depend on m unlike HL;

$$a_2(m)=-(J_{11}+J_{33})$$
$$=-\ \{md+[\psi'(u_a)+m(\frac{\psi'(u_a)}{u_a}-x\frac{1}{\sigma_a})](1-u_a)\}$$
$$=-[\psi'(u_a)+m\frac{\psi'(u_a)}{u_a}](1-u_a)$$
$$=-\psi'(u_a)(1+\frac{m}{u_a})(1-u_a)\,,^{13} \tag{35}$$

it does not depend on x and it depends on m as in HL. This function has non-positive values $(a_2(m)\le 0)$ for magnitudes of the control parameter that are lower than the first critical magnitude: $m\le \overline{m}_2=-u_a$. If $a_2(m)\le 0$ for $m\in(-\infty,\overline{m}_2]$, where $-1>\overline{m}_2=-u_a<0$, the stationary state E_c is not stable.

Proposition 2 is true so long as the inequality (36), the special case of the inequality (C-3m), holds:

$$a_1a_2(m)\ =\ (\frac{\psi'(u_a)}{u_a}-x\frac{1}{\sigma_a})\ c_2(1-u_a)X\ \psi'(u_a)(1-u_a)>\ a_0=-c_2(1-u_a)\ dv_a\ \psi'(u_a) \tag{36}$$

or

$$-(d-\psi'(u_a)\frac{1-u_a}{u_a})\ c_2X\ [\psi'(u_a)+m\frac{\psi'(u_a)}{u_a}](1-u_a)>-c_2(1-u_a)\ dX$$

and after changing sign on the left on the right sides:

$$(d-\psi'(u_a)\frac{1-u_a}{u_a})\ c_2X\ \psi'(u_a)(1+\frac{m}{u_a})(1-u_a)<c_2(1-u_a)\ dX\,.$$

We get a simpler form of the inequality (36) after eliminating common non-zero terms on the left on the right sides:

$$[d-\psi'(u_a)\frac{1-u_a}{u_a}]\ \psi'(u_a)(1+\frac{m}{u_a})<\ d\,. \tag{36a}$$

Corollary 1. The dynamics of the system (3a), (9), (10c) in the neighbourhood of its stationary state $E_c\equiv E_a$ given by (11a) are Poincaré (locally) stable for $m\ge 0$.

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[12] As $md^2=\frac{1}{\sigma_a}x(1-u_a)\,md$.

[13] As $md=\frac{1}{\sigma_a}x(1-u_a)\,m$.

Proof.

According to (35), $a_2(m) > 0$. The multipliers on the left side of (36a) are $d - \psi'(u_a)\frac{1-u_a}{u_a} > d > 0$, $\psi'(u_a)(1+\frac{m}{u_a}) < 0$ and their product is negative, i.e. lower than the positive stationary growth rate of net output on the right side. So the stationary state E_c is asymptotically stable for $m \geq 0$, q.e.d.

The closed loop control over profitability and employment ratio enables to replace perpetual growth cycles, that take place in the system (3a), (9) and (10a) for $m \geq m_0$, by damped growth cycles or by a smooth asymptotical transition to the stationary state E_c.

Corollary 2. The stronger the direct scale effect ($m > 0$), typically the faster is asymptotical convergence to the stationary state E_c in vicinity of this state.

Proof.

As the stationary state E_c is asymptotically stable all three roots of the characteristic equation has negative real parts $Re(\lambda_1) < 0, Re(\lambda_2) < 0$, $Re(\lambda_3) < 0$; the sum total of these roots equals the trace of the Jacoby matrix $\lambda_1 + \lambda_2 + \lambda_3 = -a_2(m) = \psi'(u_a)(1+\frac{m}{u_a})(1-u_a) < 0$, that has the negative derivative

$$-\frac{\partial a_2(m)}{\partial m} = \psi'(u_a)\frac{1-u_a}{u_a} < 0.$$

Due to these properties, increases in m bring about decreases in the sum total of the three roots of the characteristic equation that typically facilitates the asymptotical convergence to the stationary state E_c, q.e.d.

But unlike KGM-II, now destabilizing are negative values of the control parameter m if these values are sufficiently large absolutely. There is a violation of the condition (C-3m) at the first critical magnitude of the control parameter $\overline{m}_2 = -u_a$ since $a_2(\overline{m}_2) = 0$.

A2.2 The Andronov – Hopf bifurcation in CL

Lemma 3. The stationary state E_c loses its stability at a higher magnitude of the control parameter than $-1 < \overline{m}_2 = -u_a < 0$. It happens when the inequality (16-3m) terns into equality.

Proof.

Let this third local stability requirement turns into equality:

$$a(m) = a_1 a_2(m) - a_0 = 0$$

or

$$J_{32}(J_{23}J_{33} + J_{21}J_{13}) = 0$$

or

$$J_{23}J_{33} + J_{21}J_{13} = 0 \text{ as } J_{32} > 0.$$

Inserting the specific expressions we get

$$\left(\frac{\psi'(u_a)}{u_a} - x\frac{1}{\sigma_a}\right)X\ [\psi'(u_a) + m(\frac{\psi'(u_a)}{u_a} - x\frac{1}{\sigma_a})](1-u_a)$$

$$-\frac{d}{\sigma_a}X\ [-\psi'(u_a) - m(\frac{\psi'(u_a)}{u_a} - x\frac{1}{\sigma_a})]\sigma_a = 0,$$

whereby after dividing by $X(1-u_a)$ we get

$$\left(\frac{\psi'(u_a)}{u_a} - x\frac{1}{\sigma_a}\right)\ [\psi'(u_a) + m(\frac{\psi'(u_a)}{u_a} - x\frac{1}{\sigma_a})] + \frac{x}{\sigma_a}\ [\psi'(u_a) + m(\frac{\psi'(u_a)}{u_a} - x\frac{1}{\sigma_a})] = 0$$

or

$$\frac{\psi'(u_a)}{u_a}\,[\psi'(u_a)+m(\frac{\psi'(u_a)}{u_a}-x\frac{1}{\sigma_a})]=0$$

and

$[\psi'(u_a)+m(\frac{\psi'(u_a)}{u_a}-x\frac{1}{\sigma_a})]=0$ ➔ the second critical magnitude of the control parameter is

$$0>\overline{m}_0=-\frac{\psi'(u_a)}{\frac{\psi'(u_a)}{u_a}-x\frac{1}{\sigma_a}}. \qquad (37)$$

It does not depend on x as in HL.

Take into account inequalities (34), (35) and (36a). The linear function $a(m) = a_1a_2(m)-a_0$ is continuous on the segment $[m_2, 0]$ and changes signs at the segment ends: $a(m)>0$ for $m=0$ and $a(m)=-a_0<0$ for $m=\overline{m}_2=-u_a$ hence $0>\overline{m}_0>-u_a$.

Checking the transversality condition will complete the proof of the Andronov – Hopf bifurcation at $\overline{m}_0$. Recall $a_1 = const > 0$; $\frac{\partial a_0}{\partial m}=0$, $\frac{\partial a_1}{\partial m}=0$, $\frac{\partial a_2}{\partial m}=-\frac{\psi'(u_a)}{u_a}(1-u_a)>0$. We have the ratio that is negative

$$\frac{\partial(\mathrm{Re}\,\lambda_{2,3}(m))}{\partial m}=\frac{-\left[\frac{\partial a_1}{\partial m}a_2+a_1\frac{\partial a_2}{\partial m}\right]+\frac{\partial a_0}{\partial m}}{2(a_1+a_2{}^2)}=\frac{-(0a_2+a_1\frac{\partial a_2}{\partial m})+0}{2(a_1+a_2{}^2)}<0,\ \text{q.e.d.}^{14}$$

The period of cycle for the closed orbits near E_c is about $2\pi/\sqrt{a_1}$ or

$$T\approx\frac{2\pi}{\sqrt{[d-\psi'(u_a)\frac{1-u_a}{u_a}]c_2X}}. \qquad (38)$$

It depends neither on m_0 nor on x.

This period of the growth cycle near the stationary state is the shorter, the higher are growth rate of net output, the employment ratio and the control parameter $c_2 > 0$, the steeper is the slope of the mechanization (automation function) with respect to the relative wage. This period is the longer, the higher is the relative wage. It is independent of the stationary capital-output ratio, unlike the period of growth cycle in HL. Growth cycle of the latter type for CL is not realistic since it assumes sufficiently large direct negative return to scale contrary to reality.

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[14] There must be negative sign unlike the transversality condition in HL, as bifurcation is for m declining from 0 to $\overline{m}_0<0$.