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Stability and bifurcation of a model of the price mechanism of export-import transactions

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In modern literature on economics, there exists a rather detailed qualitative description of a mechanism of the formation of home commodity prices, based on an analysis of the dynamics of streams of gold and currency in export-import transactions [1]. A special attention is paid to such an important factor as an index of the trade balance whose surplus or deficit determine the tendency of price changes.

Before considering a formalized mathematical model of the dynamics of prices on the basis of the classical monetary theory, we must specify a number of assumptions:

1) we consider a scheme of free trade without any influence of governments and monopolistic agencies;

2) the national income level is given, whereas the level of prices is determined on the basis of a qualitative monetary theory;

3) during the considered period, changes in the money supply are determined only by a deficit (or a surplus) of the trade balance;

4) as currency exchange rates are assumed to be fixed, we can set them equal to unity, which is equivalent to international transactions in gold;

5) transport expenses, insurance and other expenses are not taken into account both for commodity and financial streams.

Below, we use the following notation: Q is the money supply; V is the speed of the money turnout (a constant); Y is the national income level (a constant); P is the level of domestic prices; P_M is the level of foreign prices (a constant); M is the volume of import; X is the volume of export.

The basic equation of the model has the form

$$QV = PY. \tag{1}$$

The function of the volume of export is a decreasing function of the domestic price

$$X = X(P), \quad \frac{dX}{dP} < 0,$$

whereas the function of import, in contrast, is an increasing function of the domestic price:

$$M = M(P), \quad \frac{dM}{dP} > 0.$$

A relation of the form

$$P^* X(P^*) - P_M M(P^*) = 0$$

determines the condition of equilibrium of the trade balance. We assume that this equation admits positive solutions that determine equilibrium values of the domestic price P^*.

A violation of equilibrium is accompanied by changes in the money supply and is expressed by means of the equation

$$\frac{dQ}{dt} = PX(P) - P_M M(P). \tag{2}$$

As follows from equation (1), a change in the money supply leads to a change of the domestic price $P(t)$. It is justified to assume that such a change is not instantaneous, that is, there occurs a time delay characterized by a constant τ. In this case, equation (1) can be written as

$$\frac{dP(t)}{dt} = \frac{V}{Y}\frac{dQ(t-\tau)}{dt}. \tag{3}$$

Equations (2) and (3) yield a difference-differential equation that describes the dynamics of the domestic price:

$$\frac{dP(t)}{dt} = \frac{V}{Y}\left\{P(t-\tau)X\left[P(t-\tau)\right] - P_M M\left[P(t-\tau)\right]\right\}. \tag{4}$$

We will assume that the functions of export $X(P)$ and import $M(P)$ are nonlinear and can be expanded into a Taylor series up to the third power, i.e.,

$$X(P) = X_0 - X_1 P + X_2 P^2 + X_3 P^3 + O\left(P^4\right),$$
$$M(P) = M_0 + M_1 P + M_2 P^2 + M_3 P^3 + O\left(M^4\right), \qquad (5)$$

where

$$X_i = \frac{d^i X(P^*)}{i! dP^i}, \quad M_i = \frac{d^i M(P^*)}{i! dP^i}, \qquad i = 0, 1, 2, 3,$$

are corresponding derivatives of the functions $X(P)$ and $M(P)$ at the point of equilibrium P^*. Let us introduce a new quantity $\bar{P}(t) = P(t) - P^*$ that has the meaning of a deviation of the domestic price from its equilibrium value. In this case, equation (4), taking into account (4), is reduced to the form

$$\frac{d\bar{P}}{d\bar{t}} = \frac{\tau V}{Y}\Big\{(X_0 - P^* X_1 - P_M M_1)\bar{P}(\bar{t}-1)$$
$$+ (P^* X_2 - P_M M_2 - X_1)\bar{P}^2(\bar{t}-1)$$
$$+ (P^* X_3 - P_M M_3 + X_2)\bar{P}^3(\bar{t}-1)\Big\}, \qquad (6)$$

where $\bar{t} = \tau t$.

At first, we investigate the conditions of local stability, restricting ourselves to the linear part of (6), i.e.,

$$\frac{d\bar{P}}{d\bar{t}} = \frac{\tau V}{Y}\alpha\bar{P}(\bar{t}-1), \quad \alpha = X_0 - P^* X_1 - P_M M_1. \qquad (7)$$

The characteristic equation for (7) is

$$\lambda - \frac{\tau V \alpha}{Y} e^{-\lambda} = 0. \qquad (8)$$

Applying the well-known results of the theory of the stability of difference-differential equations [2] to equation (7), we obtain the necessary and sufficient conditions of the stability:

$$0 < -\frac{\tau V \alpha}{Y} < \frac{\pi}{2}. \qquad (9)$$

As follows from the left-hand side of the double inequality (9), the quantity α is negative, whereas the right-hand side implies that the absolute value of α is less than $\frac{Y\pi}{2\tau V}$. Condition (9) has a rather transparent economic interpretation, which can be illustrated by a transformation of the initial parameters.

Let

$$\alpha = X_0 \left[1 - \eta_X - \eta_M\right],$$

where

$$\eta_X = \frac{P^* X_1}{X_0}, \quad \eta_M = \frac{P^* M_1}{M_0},$$

under the condition $P^* X_0 = P_M M$. We will call η_X and η_M elasticities of the functions of export and import with respect to the price. As $X_0 > 0$, the condition $\alpha < 0$ is equivalent to

$$1 - \eta_X - \eta_M < 0,$$

or

$$\eta_X + \eta_M > 1,$$

which corresponds to so-called Marshall-Lerner's conditions. At the same time, the condition (9) reduces to the form

$$1 < \eta_X + \eta_M < 1 + \frac{Y\pi}{2\tau V}. \tag{10}$$

Thus, the economic interpretation of the conditions of local stability consists in the fact that the sum of the elasticities should not only exceed unity, but should also be less than another critical value. In other words, instability in the investigated economic model may occur not only when the sum of elasticities is sufficiently small, but also in the case of a considerably large value of this sum.

Let us study the behavior properties of the initial dynamic system (6) in a small neighborhood of the bounds of the inequality (10). First, we consider the situation near the lower bound. Let us introduce a small parameter $\mu = 1 - \eta_X - \eta_M$. In this case, with changing sign of μ, an eigenvalue of the linearized problem passes through zero, and the stationary state P^* either may not exist or may split into several stationary states. That is, there occurs a bifurcation of stationary solutions.

The difference-differential equation (6) can be represented as follows:

$$\frac{d\bar{P}}{d\bar{t}} = A_1 \bar{P}\left(\bar{t} - 1\right) + A_2 \bar{P}^2 \left(\bar{t} - 1\right) + A_3 \bar{P}^3 \left(\bar{t} - 1\right), \tag{11}$$

where

$$A_1 = \frac{\tau V X_0}{Y}\mu, \quad A_2 = \frac{\tau V}{Y}\left(P^* X_2 - P_M M_2 - X_1\right),$$
$$A_3 = \frac{\tau V}{Y}\left(P^* X_3 - P_M M_3 + X_2\right).$$

We additionally assume that the quantity $\nu = \frac{P^* X_2 - P_M M_2 - X_1}{X_0}$ is also small, i.e., $A_2 = \frac{\tau V X_0}{Y}\nu$. Using the method of central manifold [3], one can show that, under the condition of smallness of A_1, A_2 and a finite delay time τ, the difference-differential equation (11) is topologically equivalent to the differential equation

$$\frac{d\bar{P}}{d\bar{t}} = A_1 \bar{P}\left(\bar{t}\right) + A_2 \bar{P}^2\left(\bar{t}\right) + A_3 \bar{P}^3\left(\bar{t}\right). \tag{12}$$

By means of a linear change of variable $\tilde{P} = \bar{P} + \frac{A_2}{3A_3}$, equation (12) is given the form

$$\frac{d\bar{P}}{d\bar{t}} = \alpha_1 + \alpha_2 \tilde{P} + A_3 \tilde{P}^3, \tag{13}$$

where

$$\alpha_1 = \frac{2A_2^3}{27A_3^2} - \frac{A_1 A_2}{3A_3}, \quad \alpha_2 = A_1 - \frac{A_2^3}{3A_3}.$$

The transformation $\tilde{P}\left(t\right) = \left|B\right| V\left(t\right)$ explicitly yields Poincaré's normal form for the differential equation (13):

$$\frac{dV}{dt} = \beta_1 + \beta_2 V + S V^3, \tag{14}$$

where

$$B = \frac{1}{\sqrt{A_3}}, \quad \beta_1 = \frac{\alpha_1}{\left|B\right|}, \quad \beta_2 = \alpha_2, \quad S = signB = \pm 1.$$

For definitiveness, we set $S = -1$. Equation (14) can have three points of equilibrium. A "fold" bifurcation is determined by a line R in the plane β_1, β_2, given by a projection of the line

$$\Gamma : \quad \begin{cases} \beta_1 + \beta_2 V - V^3 = 0, \\ \beta_2 - 3V^2 = 0 \end{cases}$$

onto the parameter plane. By eliminating V from these equations, we get the projection

$$R = \left\{ (\beta_1, \beta_2) : 4\beta_2^3 + 27\beta_1^2 = 0 \right\}.$$

The curve R is called a semicubic parabola and has two branches R_1, R_2 that meet each other tangentially at a "cusp" point given by $\beta_1 = 0$, $\beta_2 = 0$. The separatrix divides the parameter plane into two regions (see the Figure).

In region 1, in front of the boundary line, there are three points of equilibrium: two of them are stable and one is unstable. In region 2, beyond the boundary line, there is a unique equilibrium point, and it is stable. A non-degenerate "fold" bifurcation occurs by crossing R_1 or R_2 at any point in the plane of the parameters β_1, β_2 which is different from the origin. If the curve is crossed from region 1 to region 2, the right stable equilibrium point merges with the unstable one and both of them vanish. Analogously, the left stable point of equilibrium merges with the unstable one on the line R_2. When approaching the "cusp" point in front of region 1, all the three equilibrium states merge as a triple root of the right-hand side of the initial equation (14). Of importance is also the fact that, by passing from a stable regime to unstable one in (14), hysteresis phenomenon is observed and a catastrophe occurs [3].

The case $S = 1$ can be analyzed along the same lines.

Now we analyze the situation when the sum of elasticities $\eta_X + \eta_M$ is close to the value at the right bound of the inequality (10). Let us introduce a small parameter $\mu = \frac{\tau V X_0}{Y} (\eta_X + \eta_M - 1) - \frac{\pi}{2}$. In this case, the difference-differential equation (6) takes the form

$$\frac{d\bar{P}}{d\bar{t}} = -\left(\mu + \frac{\pi}{2}\right) \bar{P}(\bar{t}) + \frac{\tau V}{Y} \left\{ (P^* X_2 - P_M M_2 - X_1) \bar{P}^2 (\bar{t} - 1) \right.$$
$$\left. + (P^* X_3 - P_M M_3 + X_2) \bar{P}^3 (\bar{t} - 1) \right\}, \tag{15}$$

The characteristic equation for the linear part of (15) reads

$$\lambda + \left(\mu + \frac{\pi}{2}\right) e^{-\lambda} = 0. \tag{16}$$

Let us find out whether this equation posses a pair of pure imaginary roots $\lambda = \pm i\omega$, $i^2 = -1$, $\omega > 0$. If $\lambda = \pm i\omega$, then

$$\left(\mu + \frac{\pi}{2}\right) \cos\omega = 0, \quad \omega - \left(\mu + \frac{\pi}{2}\right) \sin\omega = 0.$$

This implies that $\omega = (2n+1)\frac{\pi}{2}$ $(n = 0, 1, \ldots)$, and $\mu = \pi n$ $(n = 0, 2, 4, \ldots)$ are critical values of μ. We will consider only the case $n = 0$. Thus, it has been shown that equation (16) for $\mu = 0$ possesses a pair of pure imaginary roots $\pm\frac{i\pi}{2}$. It is not difficult to show that (16) does not posses roots with positive real parts.

Given that λ is analytic with respect to μ, differentiation of (16) yields:

$$\left.\frac{d\lambda}{d\mu}\right|_{\mu=0} = \frac{\frac{\pi}{2} + i}{1 + \frac{\pi^2}{2}}.$$

As a results, for $\mu = 0$, all the conditions of Hopf's theorem are met, because the real part of the derivative of the eigenvalue with respect to μ is not equal to zero.

Using the above results, we will prove that the system (15) admits a family of periodic solutions $\bar{P}_\epsilon(\bar{t})$ $(\epsilon > 0)$, where ϵ is a measure of the amplitude

$$\max_t \left|\bar{P}_\epsilon(\bar{t})\right|,$$

and ϵ is sufficiently small at that.

Our task is to study the bifurcation of birth (death) of a cycle in the difference-differential equation (15). To reduce this equation to a single complex equation, we will again employ the method of central manifold [4].

Equation (15) contains a great number of parameters. In order to simplify further consideration, we make a change of variable

$$\bar{P}(\bar{t}) = \frac{X_0(\eta_X + \eta_M - 1)}{X_1 + P_M M_2 - P^* X_2} u(\bar{t}).$$

For $\mu = 0$, equation (15) takes the form

$$\frac{du(\bar{t})}{d\bar{t}} = -\frac{\pi}{2}\left[u(\bar{t}-1) + u^2(\bar{t}-1) + \gamma u^3(\bar{t}-1)\right]$$

$$+ (P^* X_3 - P_M M_3 + X_2)\bar{P}^3(\bar{t}-1)\Big\}, \qquad (17)$$

where

$$\gamma = \frac{X_0(X_2 + P^* X_3 - P_M M_3)}{(X_1 - P^* X_2 + P_M M_2)^2}.$$

By use of the theorem on central manifold, the difference-differential equation (17) is reduced to the complex differential equation

$$\dot{Z} = i\frac{\pi}{2}Z + g_{20}\frac{Z^2}{2} + g_{11}Z\bar{Z} + g_{02}\frac{\bar{Z}^2}{2} + g_{21}\frac{Z^2\bar{Z}}{2}, \tag{18}$$

where $Z = Z(t)$ is a complex-valued function, $\bar{Z}(t)$ is the complex conjugate to $Z(t)$. The coefficients of the powers of Z and $\bar{Z}$ are determined by

$$\begin{gathered} g_{20} = -g_{11} = g_{02} = \pi\bar{D}, \\ g_{21} = 2\pi\left\{\left(\frac{2-11i}{5} - i\frac{3\gamma\pi}{4}\right)\bar{D} + \frac{3}{4}D\bar{D} + i\bar{D}^2\right\}, \\ D = \frac{1+i\frac{\pi}{2}}{1+\frac{\pi^2}{4}}, \quad \bar{D} = \frac{1-i\frac{\pi}{2}}{1+\frac{\pi^2}{4}}. \end{gathered} \tag{19}$$

The availability of concrete values of the coefficients of the nonlinear part of equation (18) makes it possible to employ equations of reference [4] for the determination of the stability, the direction of birth, the period and the asymptotic form of periodic solutions of a small amplitude that realize an Andronov-Hopf bifurcation from the stationary state. From (19), we get an explicit form of Liapunov's first quantity

$$C_1(0) = \frac{\pi}{1+\frac{\pi^2}{4}}\left\{\frac{2}{5} - \frac{\pi}{2}\left(\frac{11}{5} + \frac{3\gamma\pi}{4}\right) - i\left(\frac{\pi}{5} + \frac{11}{5} + \frac{3\gamma\pi}{4}\right)\right\}. \tag{20}$$

The real parts of (20) are negative under the condition

$$\gamma > \gamma_c = \frac{16-44\pi}{15\pi^2} \approx -0.826\ldots \tag{21}$$

This means that a limit cycle will be stable for $\gamma > \gamma_c$ and unstable under a violation of the condition (21). As to the stable limit cycle, we have obtained expressions for its main characteristics:

1) the amplitude is $\epsilon = \sqrt{\frac{20\mu}{15\gamma\pi^2+44\pi-16}}$;

2) the period is $T_\epsilon = 4\left(1 + \frac{2}{5\pi}\epsilon^2\right)$;

3) the asymptotic form of the periodic solution is

$$u_\epsilon(\bar{t}) = 2\epsilon\cos\left(\frac{\pi\bar{t}}{2}\right) + 2\epsilon^2\left[\frac{2}{5}\sin(\pi\bar{t}) - \frac{1}{3}\cos(\pi\bar{t}) - 1\right],$$

$$\bar{P}_\epsilon(\bar{t}) = \frac{X_0(\eta_X + \eta_M - 1)}{X_1 + P_M M_2 - P^* X_2} u_\epsilon(\bar{t}).$$

In this case, the cycle is born in the direction $\mu > 0$, whereas the born periodic solution is asymptotically stable. The corresponding regime of the formation of auto-oscillations is called soft. However, under the condition $\gamma < \gamma_c$, we have an unstable limit cycle. The loss of stability under the formation of auto-oscillations occurs rigidly, i.e., there exists a possibility of a sudden transition to a new stationary or nonstationary regimes. In a real system, this way of the loss of stability is accompanied by a catastrophe.

The initial system (17) exhibits the most interesting behavior in the situation when the parameter γ is close to its critical value γ_c, i.e., when the quantity $\xi = \gamma - \gamma_c$ is small. In this case, we can observe the so-called Bautin's bifurcation that admits simultaneous coexistence of both the stable and unstable cycles. An analysis of qualitative properties of this bifurcation involves a Taylor expansion of the right-hand side of (4) up to fifth order. Furthermore, using the method of central manifold, one has to reduce the functional equation to a complex differential equation that contains nonlinear terms up to fifth order. After that, using the corresponding bifurcation formulas, one can evaluate Liapunov's second quantity, whose sign uniquely determines the main parameters and peculiarities of this bifurcation.

Thus, the results of our investigation of behavior properties of the difference-differential equation (4) allow us to draw the following conclusions about bifurcations of codimension two:

1) at the left boundary $(\eta_X + \eta_M = 1)$, there exists a bifurcation of the "cusp" type;

2) at the right boundary $(\eta_X + \eta_M = 1 + \frac{Y\pi}{2\tau V})$, there exists a bifurcation of Bautin's type.

The separatrices in the parameter space for each of the above-mentioned bifurcations, respectively, take the following form:

a) for the "cusp" bifurcation,

$$X_0(1 - \eta_X - \eta_M)(P^* X_3 - P_M M_3 + X_2) = \frac{1}{4}(P^* X_2 - P_M M_2 - X_1)^2;$$

b) for Bautin's bifurcation,

$$X_0(1 - \eta_X - \eta_M)(P^* X_3 - P_M M_3 + X_2).$$

$$= \frac{16 - 44\pi}{15\pi^2} \left(P^* X_2 - P_M M_2 - X_1 \right)^2 .$$

In conclusion, we point out that an application of mathematical methods to the analysis of concrete objects is related to numerical results and corresponding contextual interpretation. In this sense, the function of qualitative theory of difference-differential equations is somewhat different: It focuses attention on characteristic features of the phenomenon as a whole, on qualitative prediction of its behavior. Our task is a search for irreducible topological structures that subdivide the phase picture of the system. Applied part consists in establishing a relationship between these structures of the phase space and economic processes, complemented by implementation of a bifurcation analysis. In this case, we would have to take into account properties of the real object that impose restrictions on both the phase variables and the constants of equations (4)-(6).

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Bifurcation diagram

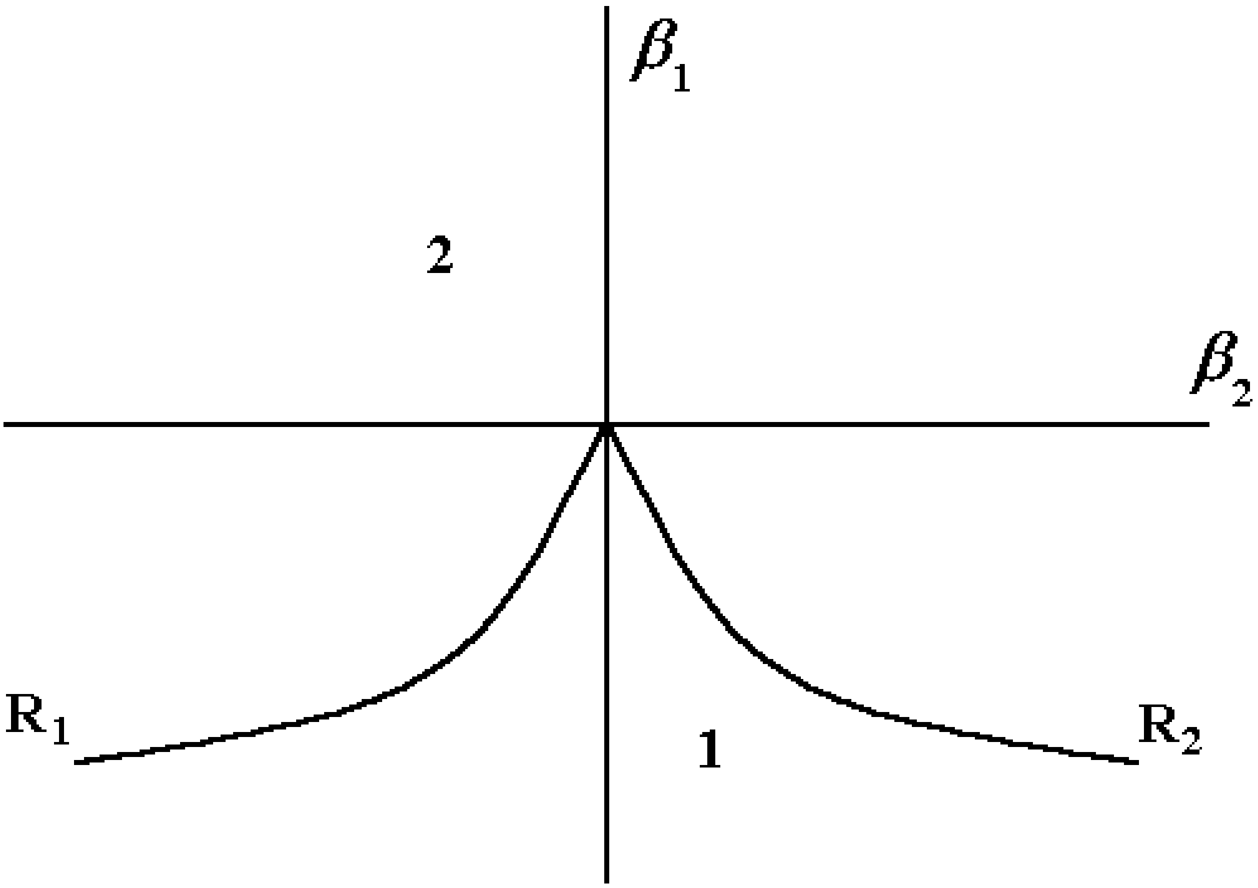

Fig.1

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