CHAOS OUT OF STIFF MODELS

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ABSTRACT

While nonlinear combinations of multiple modes existing in complex oscillatory systems may generate chaotic behavior in real systems, the studies of chaos attempted in system dynamics have often resorted to forcing simplistic models of systems to chaos. This paper illustrates how chaotic modes have been constructed through the creation of mis-specifications and anomalies in the model structure and parameters. This process has not only reduced the models to artifacts with little relevance to problem solving but has also invariably introduced a *stiff* structure that is susceptible to considerable building up error as numerical integration methods are used with long simulation times. The paper concludes that a model must qualify as an empirically valid system by meeting the requirements of the normal system dynamics practice if the chaotic modes it generates are to be of practical value.

Key Words: Deterministic Chaos, Social Systems, System Dynamics, Models, Computer Simulation

INTRODUCTION

The existence of coupled major negative feedback loops together with the nonlinear relationships often found in real systems may give rise to many complex nonlinear combinations of multiple oscillations of different periodicities. In cases when these periodicities are non-converging, some of these combinations can be so complex that their envelope may never seem to repeat while the relationship between successive cycles within the envelope appears to have no perceptible order. The behavior so created falls within the definition of deterministic chaos [Andersen 1988].

Unfortunately, in system dynamics literature chaos has been treated largely as an artifact. In order to create chaos, Researchers have often used simplistic models with unrealistic and often *stiff* structure which not only violates the normal modelling heuristics of system dynamics but is also susceptible to much building up error error when numerical integration methods are used with long simulation times (Kreyszig 1972). The evidence of chaos in real world data is both limited and inconclusive. Even when asymmetric modes can be observed, they may be interpreted both as auto-correlated noise or chaos depending on the way the underlying processes are modeled [Chen 1988]. Since chaotic modes exhibited by the models appear only with certain parameter values and exogenous inputs lying within narrowly specified ranges that are susceptible to integration errors, the relevance to the real-world systems of the chaotic behavior appearing in the models remains unclear; nor has experimentation with them to-date evolved any principles for system improvement [De Greene 1990].

This paper demonstrates how complex combinations of non-linear periodicities may create complex envelops, and how stiffness may corrupt such behavior. It also reexamines the experimentation carried out earlier by the authors with chaotic models selected from the literature to show that these models incorporated both unrealistic structure and *stiffness.*

Thus chaotic modes exhibited by the models might then be unrelated to the real world as well as corrupted by building up error [Saeed and Bach 1990, 1991].

It is suggested that the treatments of chaos as an artifact, though interesting, are irrelevant to the traditional system dynamics objectives of unification of knowledge and policy design for system improvement [Forrester 1987] because of the weak integration of the chaotic models with the real world and their susceptibility to integration error. These deficiencies will have to be overcome if the research on chaos is to be of practical value.

2. CHAOS AS A COMPLEX MODE OF BEHAVIOR

The real-world systems contain many adjustment paths appearing as coupled major negative feedback loops, each creating an oscillatory mode of a given periodicity. There may also exist many nonlinear relationships governing the flows associated with the stocks in each feedback loop. Thus, there are many possibilities of creating complex nonlinear combinations of multiple frequencies leading to infinitely long envelopes with unrecognizable relationships between the successive cycles — a mode of behavior referred to as chaos.

Since multiple adjustment paths and nonlinear relationships are quite pervasive in human systems, the existence of chaotic modes in real world social phenomena cannot be ruled out. This can be demonstrated by modulating systematically the behavior of a simple linear workforce-inventory system generating undamped oscillations of a single period by an exogenous periodic function. The model is developed and simulated using iTHINK 2.0 with Runga-Kutta-4.[1] Figure 1 shows the simple linear model of a workforce inventory system used in our experimentation. In the basic version of this model, workforce adjustment depends on the inventory discrepancy while production rate is determined by the workforce. Shipments in the final version of the model used in a subsequent experiment are also constrained by a nonlinear function representing the inventory limitation.

The exact solution for inventory and workforce for the basic model is of the following form:

$$f_t = A\cos\Omega_0 t + B\sin\Omega_0 t$$

where A, B and Ω_0 are functions of the system parameters. When disturbed by a step change in shipments, this system will generate sustained oscillations.

A forcing function is now applied to this system in the form of an oscillatory disturbance in the parameter representing Productivity. The forcing function is of the following form:

$$\text{Productivity}*(1+C\cos\Omega t)$$

When the Ω_0 and Ω are non-integer and non-converging numbers, the complex oscillatory pattern generated by this system will have an envelope with an infinitely long period, which

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[1] iTHINK is is a trade mark of High Performance Systems, 45 LYme Road Suite#300, Hanover, NH 03755, U. S. A.

might often be the case in reality. A phase plot obtained from this system appears at Figure 2(a). Care has been taken to assure that no stock in the defined system assumes negative values over the course of the simulation even though the inventory limitation creating a first order control on the shipments has not yet been applied. The pattern shown in Figure 2(a) is, however, not strictly chaotic since a systematic relation appears to exist between the various cycles within the envelope, leading to a discernible order in the pattern generated.

Figure 1: *Flow diagram of a simple workforce inventory system forced with an exogenous cyclical function*

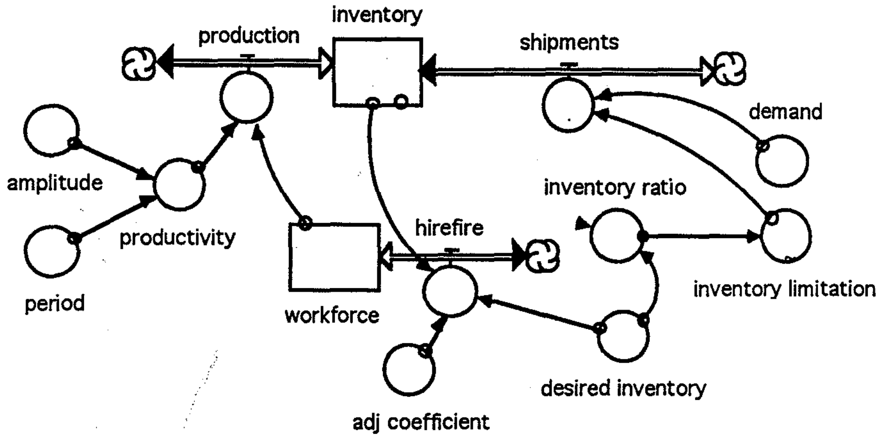
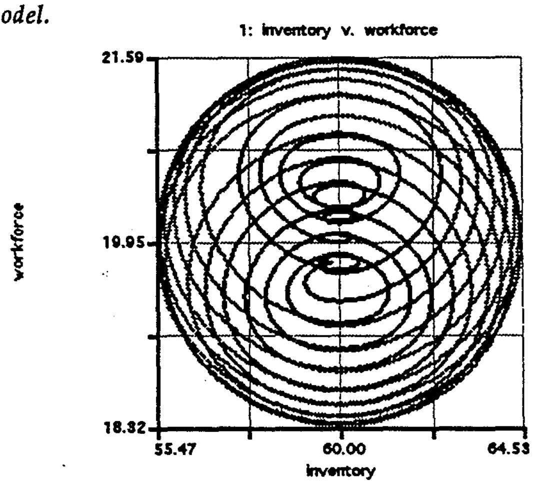


Figure 2(a): *Phase plot of workforce and inventory showing the complex pattern created by the model.*



When this pattern is further modulated by introducing a gradually sloping nonlinear first order control function representing the inventory limitation on the shipments, which also creates some stiffness in the system by introducing a variable time constant in the stock of inventory, the pattern appears to turn chaotic, as shown in Figure 2(b). The inventory limiting function applied is also shown in Figure 2(b). The behavior in Figure 2(b), although chaotic at first, tends to converge to a limit cycle. However, when the inventory limiting function is made steeper, which also increases the degree of stiffness of the model by creating sudden changes in the time constant of the stock of inventory, the converging characteristic disappears and the system appears to display sustained chaotic behavior, as shown in Figure 2(c).

Figures 2(b): Modified pattern when a gradual inventory limitation is introduced

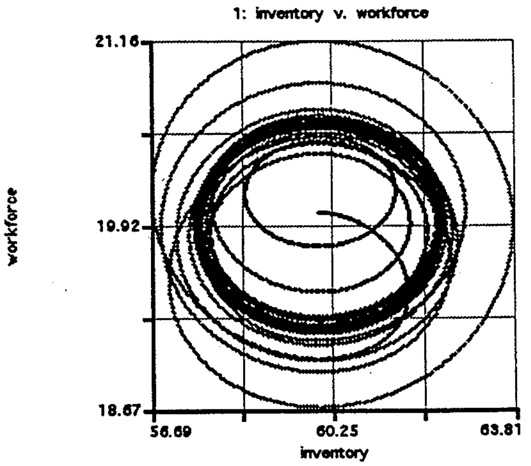
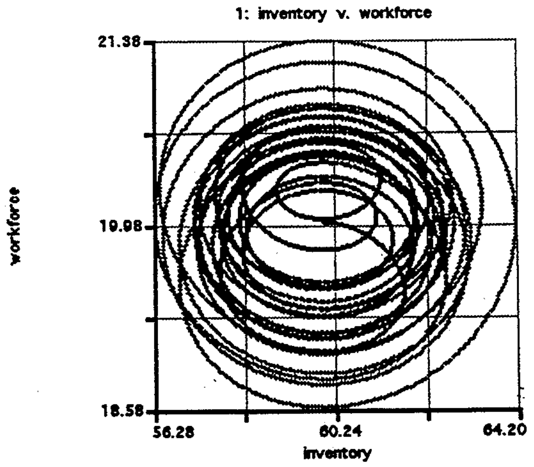


Figure 2(c): Modified pattern when a steep inventory limitation is introduced



Two issues are to be examined here. First, which of the nonlinear functions applied is a reasonable approximation of the reality? Second, what contributes to the chaos, building up error resulting from numerical integration of a stiff model or the creation of a non-converging envelope?

As for the first issue, both functions can be defended, depending on the level of aggregation used in describing the stock representing inventory. If this stock contains many types of widgets in sum, a slight shortage in the aggregate could mean a stock out for certain types of widgets, which would result in turning away a significant number of customers. Thus, a gradually sloping function would make sense. On the other hand, if inventory represents only one type of widget, only a high level of shortage would turn away a significant number of customers, and the steeper second type of function would be more appropriate.

The issue of the source of chaos cannot, however, be settled easily. While there is a possibility that the nonlinear combination of the two frequencies has generated a non-converging and complex envelope associated with chaos, there is also ample reason to surmise that the steep non-linear functions creating sudden changes in time constants have interfered with the numerical integration process used. It is widely recognized that a high degree of stiffness resulting from the presence of steep nonlinear functions in a model can create significant building up error even when sophisticated integration methods such as Runga-Kutta-4 as used, which could be responsible for the observed erratic behavior identified as chaos. Firstly, the computer round off error contributes to the building up error since the integration interval must remain quite small during the long simulation times used in the study of chaos. Secondly, since the derivatives change fast for the steep functions, the local approximation error becomes significant and this leads to selection of inappropriate integration intervals, which amplifies integration error. The two types of errors create considerably large building up error [Kreyszig 1972, Pugh-Roberts Associates 1986]

It is quite difficult to say which source is dominating the chaotic patterns shown in Figures 2(b) and 2(c); an infinitely complex envelope or the building up error created over the course of numerical integration. It can be said, however, that the contribution of integration error is higher in Figure 2(c) than in Figure 2(b), since the former incorporates a steeper nonlinear function. The reader should be reminded here that the former case also displays sustained chaos.

THE TREATMENT OF CHAOS IN SYSTEM DYNAMICS LITERATURE

The authors have carried out extended experimentation with selected system dynamics models of social phenomena recently used in the studies on deterministic chaos through computer simulation. The results of this experimentation were reported in the proceedings of the 1990 International System Dynamics Conference [Saeed and Bach 1990]. All experiments discussed in Saeed and Bach (1990) were performed using Professional DYNAMO Plus program with Runga Kutta-4.[2] Five well-known models were selected for experimentation. These included the Waycross and Weidlich models of migratory dynamics discussed in Rasmussen and Mosekilde 1988, Mosekilde, et. al (1985), Reiner et. al. (1988) and Richardson and Sterman (1988); two versions of a model of resource allocation in a firm shown to display chaotic modes respectively by Mosekilde, et. al.(1988) and Andersen and Sturis (1988), and a simple model of the economic long wave originally developed by Sterman (1985) and shown to display chaos in Rasmussen, et. al. (1985).

The Waycross and Weidlich models incorporate the same causal structure, representing the

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[2] Professional DYNAMO Plus is a trade mark of Pugh-Roberts Associates, Five Lee Street, Cambridge, MA 02139, U. S. A.

migratory dynamics of multi-ethnic communities resulting from the imbalances between the two population groups in three adjacent neighborhoods. However, the former model has a very steep first-order control on the outflows from the population stocks in the three neighborhoods, while the latter incorporates a very steep function representing the effect of the population imbalance between pairs of neighborhoods on migratory flows. Both formulations are not only unnecessarily complex and unrealistic, they also create considerable stiffness in the model. Minor corrections creating reasonable measures of imbalance and first-order control eliminate the chaotic modes in these models, replacing them with limit cycles.

Anderson and Sturis (1988) and Rasmussen and Mosekilde (1988) use the same model of resource allocation in a firm to produce chaotic modes, although they refer to the model differently and use slightly differing parameters and slopes of non-linear functions. The model deals with resource allocation between production and sales activities in a firm whose total resources are fixed. Product availability is treated in the model as a function of the product inventory only, rather than of the inventory coverage that takes into account both supply and demand. Customer loyalty is then modelled as a very steep non-linear function of availability. This unrealistic and stiff structure creates chaos which disappears either when availability is reformulated as inventory coverage, or when the nonlinear function representing customer loyalty is made less steep. Both these changes make the model realistic as well as less stiff.

Sterman's model of long wave contains a simple and generally robust structure representing an aggregate production sector that orders capital from itself according to required production capacity. For normal parameter values, the model exhibits a characteristic limit cycle. It produces chaotic behavior when one of its behavioral functions is made extremely steep and an unrealistically high cyclical exogenous disturbance is applied [Rasmussen et. al. 1985]. This chaotic mode disappears when the amplitude of the exogenous disturbance is decreased, or when the slope of the questionable behavioral function is reduced to a realistic value, both measures also reducing stiffness.

We have traced the occurrence of chaos in the experimented models to the four types of modelling errors summarized in Table 1. These are: non-robust rate equations; an unrealistic decision information basis; an unrealistic order of magnitude of response to information; and excessive exogenous disturbance. We have now further discovered that each modelling error also made the models excessively stiff. Minor changes in the experimented models, which improve their correspondence to reality while simultaneously reducing stiffness, eliminate chaotic modes.

CHAOTIC MODES AND SYSTEM DYNAMICS MODELLING HEURISTICS

Our experimentation suggests that chaotic behavior appears in the models used in the literature because of mis-specifications and errors in the model formulation and, possibly, also because of stiffness. Both problems can be easily avoided by following normal system dynamics modelling heuristics. Figure 3 illustrates the widely practiced, although informally implemented, modelling heuristics recommended for system dynamics modelling work. Empirical evidence is the driving force both for delineating the micro-structure of the model and for verifying its behavior, although the information on the behavior may reside in the historical data and that concerning the micro-structure in the experience of people [Forrester 1980].

Table 1 : Pattern of modelling errors in experimented models

		Sources of Chaos			
	Models	Non-Robust Rate Equation Formulation	Unrealistic Information Basis	Unrealistic Response to Information	Excessive Exogenous Disturbance
1.	Migratory Dynamics: Waycross/ Weidlich	*			
2.	Business Policy: Rassmussen/ Andersen		*		
3.	Macro-Economics: Sterman, Long Wave			*	*

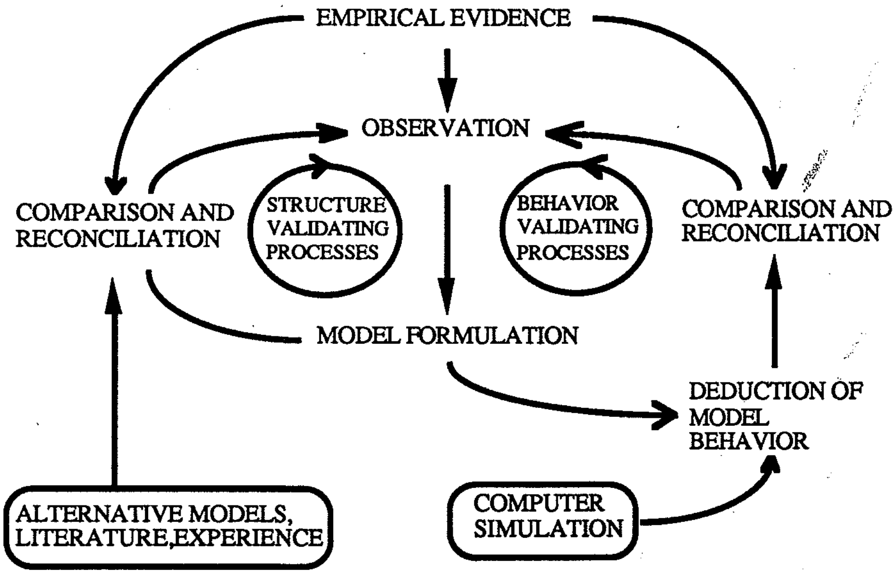
Source: Saeed and Bach(1990)

The first requirement of the method is to organize historical information into what is known in the jargon as "reference mode." The reference mode leads to the formulation of a "dynamic hypothesis" expressed in terms of the important feedback loops existing between the decision elements in the system that create the particular time variant patterns contained in the reference mode. The dynamic hypothesis must incorporate causal relations based on information on the decision rules used by the actors of the system, and not on correlations between variables observed in the historical data.

A formal model is then constructed incorporating the dynamic hypothesis along with the other structural detail of the system relating to the problem being addressed. To have credibility, the model structure must be "robust" to extreme conditions and be "identifiable" in the "real world," where the real world consists both of theoretical expositions and experiential information. A model might undergo several iterations in order to achieve an acceptable structure.

Once a satisfactory correspondence between the model and the real-world structure has been reached, the model is subjected to behavior tests. Computer simulation is used to deduce time paths of the variables of the model, which are reconciled with the reference mode. If a discrepancy is observed between the model behavior and reference mode, the model structure is re-examined and, if necessary, modified. In rare cases, such testing might also unearth missing detail concerning the reference mode, leading to a restatement of the reference mode, although for most cases, the reference mode delineated at the start of the modelling exercise must be regarded as sacred. When a close correspondence is achieved simultaneously between structure of the model (including its parameters) and the theoretical and experiential information on the system, and also between the behavior of the model and the empirical evidence about the behavior of the system, the model is accepted as a valid representation of the system [Bell & Senge 1980, Forrester & Senge 1980, Richardson & Pugh 1981].

Figure 3: *System dynamics Modelling Process*



When dealing with stiff real systems, the normal heuristics of system dynamics practice guard against the creation of a stiff model by separating the short-range and longe-range dynamics and modelling them independently (Saeed 1991). A single model must incorporate the integration processes with medium-range time constants represented as stocks, those with relatively small time constants as auxiliaries, those with relatively long time constants as constant parameters to avoid stiffness. The normal heuristics of system dynamics practice also require that, to preserve the integrity of a model and maintain the dominance of its internal trends, outside disturbances should be kept small so that they do not overpower the forces embodied in the model structure. Models with very steep functions, and parameter sets creating excessive stiffness, or those driven by powerful exogenous cyclical functions, therefore violate standard system dynamics modelling practice and should be viewed as artifacts with no real-world problem-solving relevance.

The study of chaos as an artifact forced out of models that violate the normal modelling heuristics of system dynamics is quite meaningless. The appearance of chaotic modes in such models may often signal the existence of anomalies in the model, calling for a revision of its structure and parameter specifications to improve its correspondence with reality and also to minimize any experimental error created in working with it.

WHAT SHOULD RESEARCH ON CHAOS SEEK?

We do not rule out the possibility that chaotic modes exist in real-world social phenomena when all system relations are assumed to be deterministic. The existence of memory and nonlinear responses to information in human systems may give rise to new weighting functions for the repetitive decisions taken, although a pattern might exist in the roles the human actors play on a long-term basis, which can create chaotic modes. However, notwithstanding the many learned attempts to create chaotic modes with models of physical, biological and social phenomena, we are of the view that experimentation with models alone without reference to reality and without a specific policy focus, is more alchemy than life science. The study of chaos as an artifact will often force the system to chaotic modes through subjective adjustments in the model without justifying relevance to reality, which would create mis-specifications and anomalies in the system structure and parameters, yet without creating any practical insights.

To be of practical value, research into chaos must concentrate on addressing the issue of the relevance of chaotic models to real-world phenomena and on policy design for system improvement. To accomplish this, evidence of chaos must be sought in real-world data and in realistic models of systems which also have realistic parameter sets. There has been some progress in that direction in the recent work explaining noisy physical and biological phenomena (Olson and Schaffer 1990, Mosekilde 1990), although still without a clear policy focus. As for social systems, Sterman has reported the occurrence of chaos in a model representing a multi-tier market system embodied in a game, when parameters related to the behavior of a significant minority of the subjects playing the game (20%) were used. This minority response to the decision-making information given to them was more aggressive than for the majority whose parameter set produced stable behavior (Sterman 1988). Although people acting in a gaming situation may not act realistically, and the models given to them may also not fully embody the real process they abstract, such experimentation may provide both realism and a policy focus in the treatment of chaos.

CONCLUSION

Chaotic behavior in many of the models discussed in the literature appears to arise from modelling errors and from the problems of numerical integration methods, giving the impression that chaos might be an artifact related to models and numerical integration. This impression is, however, a function of inappropriate research designs that have focussed on chaos as an artifact, often forcing the models of systems to chaotic modes through the creation of mis-specifications and anomalies in their structure and parameters, without seeking practical insights. We are uncertain how such studies of chaos can be related to the objective of real-world problem-solving, which social science in general and system dynamics in particular seeks to accomplish.

The existence of coupled major negative feedback loops together with the nonlinear relationships often found in real systems may give rise to many complex nonlinear combinations of multiple oscillations of different periodicities. Some of these combinations can be so complex that their envelope may never seem to repeat, while the relationship between successive cycles within the envelope appears to have no perceptible order. The behavior so created falls within the definition of deterministic chaos. Experimentation with simple relationships appears to confirm this point. Thus chaotic modes may exist in real systems. However, to be of practical value, the research into chaos must deal with empirical information and realistic models of real-world systems, with the aim of establishing the relevance of chaos to real-world phenomena and to policy design for system improvement, and not on the achievement of chaotic modes as artifacts from unrealistic models.

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