CHAOS STUDY IN SYSTEM DYNAMICS

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ABSTRACT

Based on our study of synergetics and dissipative structure theory, by
means of mathematics, in terms of the viewpoint of system dynamics,
this paper concentrates on the study of chaos in system dynamics: we
analyze the nature of chaos phenomenon and the characteristics of
system dynamics, put forward the viewpoint that chaos testing should
be included in model testing of S.D.; we investgate several necessary
conditions of chaos; we creat a model to question a famous sufficient
condition of chaos; then, we shed some light on the way toward which
chaos will occur. At last, we successfully apply our theoretic study
to a standard nuclear spin generator model.

1. CHAOS AND SYSTEM DYNAMICS

System dynamics is a powerful tool to study the complex social and
economic system. This complexity results from nonlinearity, high-order
and multy loop of systems. One character of such complexity is that,
though the system is deterministic, at certain point, its behavior is
like stochastic. Apart from this, the systme is extremely sensitive to
the inital value. This strong sensitivity makes two trajectories which
are very close in the beginning become quite different from each
other in time. That is to say, the system will show chaotic behavior.

To a nonliear dynamicist, a chaos system is one on which long-term
predication of the system's state is impossible because the
omnipresent uncertainty in detemining its initial state and because
the error of initial value grows exponentially fast in time.

So, it is of great significance to study chaos characteristics of S.
D. models in order to make the model be more reliable.

Fortunately, System Dynamics has paid lots of attention to model
testing. So, we recommend that chaos testing be included in
sensitivity analysis while we do model testing.

Now, let us see the mechnics of chaos.

First, we classify all macro states in the world into Table 1.

Table 1.

state type	in-between state	final state
deterministic	stable unstable	a. equilibrium b. periodical c. quasiperiodical d. tend to limitlessness e. sometimes limited sometimes limitless f. always limited
stochastic		

What classcial theories study are the stable processes, namely, process a, b, and c; process d and e are divergent; process "f" is what we call chaos:

Def. 1. A state which is always limited and which is not equilibrium, not periodical, and not quasiperiodical is named chaos.

From the above discussion, we easily get:

Theorem 1. unstable, aperiodical, bounded and extremely sensitive to inital value are the necessary conditions of chaos.

In the following, we will concrete theorem 1.

For generality and transfer ability, this paper focuses on system (1):

$$dX/dt = P(X, Y, Z)$$
$$dY/dt = Q(X, Y, Z) \quad \text{------------------- (1)}$$
$$dZ/dt = R(X, Y, Z)$$

Set q = (X, Y, Z), then, system (1) becomes

$$dq/dt = N(q) \quad \text{------------------- (2)}$$

2. BOUNDED

Def. 2. If existing a costant M, q(t) is the solution of system (2), we have:

$$\lim_{t \to \infty} \sup q(t) < M \quad \text{------------------- (3)}$$

Then, we call system (2) is dissipative ([1]).

Theorem 2. Suppose q(t) = (X(t), Y(t), Z(t)) is the solution of system (1), if (q(t), dq(t)/dt) < 0, then, system (1) is dissipative.

Proof: take

$$S*S = X*X + Y*Y + Z*Z \quad \text{------------------- (4)}$$

then (q(t), dq(t)/dt) < 0

<======> X * dX/dt + Y * dY/dt + Z * dZ/dt < 0
<======> S * dS/dt < 0   (since S > 0)
<======> ds/dt < 0

So, system (1) is dissipative.

Example 1.
$$dX/dt = - bX + Y$$

$$dY/dt = - X - bY (1 - k Z) \qquad \text{------------} \quad (5)$$
$$dZ/dt = b [a (1 - Z) - k Y*Y]$$
where a, b,k > 0. We can verify (q, dq/dt) < 0   ([2]).

Based on theorem 2, we can construct a kind of systems which are dissipative.

Theorem 3. In system (1), if P,Q,R satisfy:

    a)   dq/dt = - A(q) + G(q) + H
    b)   (q, A(q)) > 0
    c)   (q, G(q)) = 0
    d)   G(0) = 0
then (q, dq/dt) < 0.(When q is big enough)

3. UNSTABLE

In order to study unstableness, we introduce Lyapunov exponent.

Def. 3. Suppose dq(t)/dt = L(t) q(t)   ------------   (6)
    L(t) = (aij(t)), if existing B > 0 ∋ sup /aij(t)/ < B
then, for the solution vectory qi(t) of system (6), we define:
$$Zi = \lim_{t \to \infty} \sup ((1/t) * \ln qi(t)) \qquad \text{------------} \quad (7)$$
Zi are called generalized characteristic exponents. One special case of generalized characteristic exponent is so-called Lyapunov exponent.

Suppose qo(t) is a solution of system (1), sq(t) is a disturbence, then,
$$q(t) = qo(t) + sq(t) \qquad \text{---------------------------} \quad (8)$$
Put (8) into (2):
$$dqo(t)/dt + dsq(t)/dt = N(qo(t) + sq(t))$$
$$\Longleftrightarrow dqo(t)/dt + dsq(t)/dt = N(qo(t)) + (dN(qo(t)/dt*sq(t) + o(sq(t)))$$
$$\Longleftrightarrow dsq(t)/dt = [dN(qo(t)/dt)*sq(t)] \qquad \text{------------} \quad (9)$$

Def. 4. The generalized characteristics exponents of equation (9) are called Lyapunov exponents.

Haken in [3] asserts that, if the three Lyapunov exponents of system (1) are (+, 0, -), then, system is chaotic. In our study, we find this assertion is not correct.

Theorem 4. If qo(t) = qo then lyapunov exponents are the eigenvalues of N (qo).

Proof: Since qo(t) = qo , N(qo) is a constant matrix. So, equation (9) is a linear diffential equation.

Suppose A1,A2,A3, are the three eigenvalues of equation
 /AE-N(qo)/ = 0, V1, V2, V3 are relative eigenvectors. since
AiEXP(Ait)Vi=N (q0)EXP(Ait)Vi<===>N (q0)Vi=AiEVi<===>(N (q0)-AiE)Vi=0

then, EXP(A1*t)V1,EXP(A2*t)V2,EXP(A3*t)V3 are solutions of (9).

From Def. 3,
$$Bi = \lim_{t \to \infty} \sup \{1/t * \ln /\exp(Ai*t)Vi/ \}$$
$$\lim_{t \to \infty} \sup \{(Ai*t)/t + \ln /Vi/ /t\} = Ai$$
So, Lypunov exponents = eigenvalues.
where /C/ is the absolute value of C.
Example 2

$$dX/dt = - bX + Y$$
$$dY/dt = - X - bY + bkYZ \qquad ------------------ \quad (10)$$
$$dZ/dt = b - cZ - bkY*Y$$

b , c1 , k > 0. We have proved that system (10) is dissipative.

Theorem 5. C1 = (0, 0, b/C), C2,3 = {±(1/b*b*k)*SQR[(b*b*b*k-c(1+b*b))/b],±(1/b*)k*SQR[(b*b*b*k-c(1+b*b))/b],(1+b*b)/b*b*k}

are three equilibrium points of system (10).

Theorem 6. Take C = (b*b*b*k)/(1+b*b)   ------------- (11)
    ==>   A1=0   And   A*A+(b-1/b+c)*A+(bc-c/b)=0,A2,3=0.5[(1/b-b-c)±SQR[(b-1/b+c)*(b-1/b+c)-4(bc-c/b)]
for b <1,==>A2*A3<0 then, Lyapunov exponents of system (10) are
(1/b-b , 0, -c) = (+, 0, -). So, system (10) satisfies Haken's conditions. It should be chaos. But our study finds that system (10) has periodical solutions and it is not sensitive to intial values ([2]).

4. APPLICATION

Example 2 shows that it is of little possiblity to be chaotic directly at equilibrium points. Chaos is a dynamic process. The probable way toward which chaos may occur is:
    a) change parameters to make equilibrium points be limited cycles.
    b) go on changing parameters to let limited cycles bifurcate into cycle 2,4,8 ......
    c) at last, go to chaos.
In step a, we have:
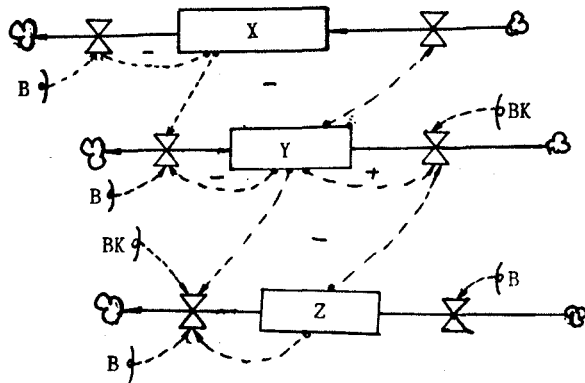Theorem 7. Suppose W is a open set, 0 beiongs to W, W is included in
      R*R*R,
f: W*(-ao,ao) ---> R*R*R
  dX/dt = f(x,a)   -------------- (12)
f(0, a) = 0, for any "a"
The eigenvalues of Df(0, 0,) are +ib(0),-ib(0),S(0),S(0)<0,
   b(0) > 0, when   a=0, (0, 0, 0,) is stable, a>0.
       (0,0,0) lost stable.

Then, for small enough a, equation (12) has a stable closed trajectory near (0, 0, 0).

Go along this road, we delve into a nuclear spin generator model.
Example 3. (system 13)

This is a three levels, seven loops, nonlinear S.D. model. It comes from nuclear physics ([4]).

Theorem 8. Example 3 is dissipative.

Theorem 9. when b = 1, k = 22, example 3 is chaos.

Our result are:
1. In example 1, we have proved that system (5) is dissipative.
2. C1 = (0, 0, 1); C2,3 = (±SQR[(k-1)*b*b-1]/k*b*b, ±SQR[(k-1)*b*b-1]/k*b, (1+b*b)/k*b*b)
   are the equilibrium points of system (5).
3. At point C=(0,0,1)

$$M = \begin{bmatrix} -b & 1 & 0 \\ -1 & -b(1-k) & 0 \\ 0 & 0 & -b \end{bmatrix}$$

   ====>A1=-b<0, A2,3=0.5[b(k-2)±SQR(b*b*k*k-4)]
  A1,A2,A3 are Lyapunov exponents, or eigenvalues.
   a) k < 2, (k - 1)b*b- 1 < 0 ====> Re(Ai) < 0, i = 1, 2, 3
At this, C1 is stable, C2,3 have not meaning yet.
   b) k < 2, (k - 1)   - 1 > 0 ====> C have got meaning. A2*A3 < 0.
   Proof: notice SQR(b*b*k*k-4) > /b(k-2)/
                       <====> (k-1)*b*b-1 > 0
   c) k > 2, (k-1)*b*b - 1 < 0 ====> A2,A3 > 0.
   d) k > 2, (k-1)*b*b - 1 > 0 ====> A2*A3 < 0.
   Under the condition of (k-1)*b*b -1 > 0, C2,3 have meaning; the
eigenvalue of C1 are (-, +, -), (-, -, +).

4. At C2,3,

$$M = \begin{bmatrix} -b & 1 & 0 \\ -1 & 1/b & y1 \\ 0 & y2 & -b \end{bmatrix}$$

   y1 = +SQR( )
   y2 = -2SQR( )
   ====> u1 = -b, u2,3=1/2{(1/b -b)±SQR[(b-1/b)(b-1/b)-8*[(k-1)*
                        b*b-1]}
According to theorem 6, take b = 1, then
       u2,3 = ±SQR2*(SQR(k-2))i,   (k > 2)
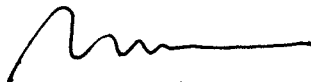At this time, at C1,A2,3 < 0; at C2,3, u2,3 = ±{SQR[2(k-2)]}i

5. At b = 1,
       dA/db = ∓[4(k-1)]/SQR[2(k-2)]
Therefore, this is a bifurcation point. So, go on.

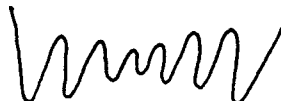Based on the above analysis, we simulated system (13):

   a) b = 0.7, k = 1.5 ====>

   b) b = 2/3, k = 3.0 ====>

   c) b = 4.0, k = 1.1 ====>

   d) b = 0.5, k = 6.0 ====>

......
When b = 1, k = 22   ===> Chaos.
In plactice, people do find that the nuclear spin generator system will
show irregular behavior. But they can not explain this phenomenon. now
our chaos study successfully shed light on it.

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