A Principle on Structure-Behavior Relations in System Dynamics Models

Burak Güneralp

University of Illinois at Urbana-Champaign
Department of Natural Resources and Environmental Sciences
S. Goodwin Ave. Urbana IL 61801 USA
Telephone: ++1 217 333 93 49
Fax: ++1 217 244-3219
guneralp@uiuc.edu

Abstract

Can a negative feedback loop take part in the generation of exponential growth? This study examines such questions and consequently presents a principle regarding the roles of feedback loops in the unfolding of system behavior for second-order linear models. In general, uncovering system structure-behavior relation is crucial in understanding the functioning of a system. In this paper, using the eigenvalue elasticity analysis, it is shown that associating loops with certain behavior modes based solely on their polarities is misleading. Six linear second-order models with similar structures are used as examples in the analysis. The models consist of three feedback loops. The resulting principle suggests that the relative locations and magnitudes of feedback loops have more significance than their polarities in the generation of system behavior. The principle may seem to state the obvious for some readers; however, its significance is its reliance on a concrete analytical analysis. There is potential for the formulation of more such principles especially for higher-order systems.

Background

Understanding the relation between structure and behavior is a demanding task, even for slightly complex models. In general, the insight on this elusive relation comes only after years of experience working with several system dynamics models. Apart from the time required for such expertise to accumulate the experienced modeler who gains the insight most of the time remains unable to properly communicate it. Graham (1977) has done an excellent job in organizing a set of principles, which helps to understand and communicate the experiences and insights on the relation between structure and behavior of systemic dynamic models. Surprisingly enough, there has been virtually no such endeavor following his line of work (but see Forrester 1983; Richardson 1995). Such principles demonstrate the inherent difficulties in inferring the behavior of even the simplest linear systems. More importantly though they pave the way to better understand and appreciate the structure-behavior relation in more complex systems.

The concept of feedback essentially stems from the notion that endogenous sources are responsible for the creation of a system's behavior. In spite of the guiding principle that the system behavior is generated as a result of the interaction between various feedback mechanisms of the system, the accurate depiction of the relation between a model's structure and its behavior has mostly remained a mystery. "Understanding model behavior" has claimed the first rank in a list of eight problem areas put forward as currently deserving the attention of system dynamics practitioners (Richardson 1996). It should be noted though that the number of related studies seems to be growing lately (Richardson 1995; Davidsen 1991; Ford 1999; Saleh and Davidsen 2000, 2001; Oliva 2003; Modjahedzadeh *et al.* 2004). As a result, there have been some improvements in revealing the structure-behavior relation through analytical and empirical means.

Can a negative feedback loop play a role in the generation of exponential growth? This study is inspired by the coincidental observation of such phenomenon by the author. The observation led to a study, which produced a principle on the structure behavior relation for second-order linear models based on an analytical examination. The eigenvalue elasticity analysis is used in the study (Forrester 1982). The next two sections briefly explain the eigenvalue elasticity analysis in the identification of dominant loops and the generic model structure used in the study. Next, the contributions of feedback loops on the system behavior are examined using the eigenvalue elasticity analysis. Then, the results are proven analytically for each model one by one. Finally, a principle on the role of loops on system behavior is stated followed by concluding remarks.

Eigenvalue elasticity analysis

The link gains characterizing the structure of a model can be related to an eigenvalue of that model using eigenvalue elasticities. The partial derivative of an eigenvalue with respect to a link gain gives the sensitivity of that eigenvalue to an infinitesimal change in that gain. The elasticity value (elasticity, in short) is defined as the sensitivity of the eigenvalue to the link gain normalized for the size of the gain and the size of the eigenvalue (Eq. 1). Thus one obtains the elasticity, which is a dimensionless measure. This enables the comparison of the elasticities of various links with each other making it a convenient measure of the relative significance (dominance) of links to a certain mode of behavior. For instance, a larger elasticity means behavior is more sensitive to a certain percentage change in one gain than another. The partial derivative of the system eigenvalues with respect to a link gain is most easily calculated by calculating the eigenvalues before and after a small change in that gain.

$$\varepsilon_{ipq} = \frac{\partial \lambda_i}{\partial a_{pq}} \frac{a_{pq}}{\lambda_i} \tag{1}$$

where ∂ ≡ partial derivative sign

λ_i ≡ i^{th} eigenvalue of the system (scalar)

a_{pq} ≡ gain of the link from variable p to q

ε_{ipq} ≡ elasticity of the eigenvalue λ_i with respect to the link gain a_{pq}

Dominant loop identification

The causal structure of a linear or linearized model can be represented as a system matrix. Each non-zero element of the matrix stands for a causal link in the original model preserving the original structure. If element a_{pq} is non-zero this means the rate of change in the state variable x_q depends on the value of the state variable x_p. An example of a linear model structure is given in Figure 1. Calculating the eigenvalue elasticity of a behavior mode with respect to all non-zero elements of the system matrix identifies the dominant structure generating that behavior (Eq. 1). The elasticities may be complex numbers. If this is the case, they refer to cyclic behavior modes and the real part gives the effect on damping ratio, while the imaginary part gives the effect on the natural period. The magnitude of the elasticity gives the overall sensitivity of the cyclic mode to a structural link (Saleh and Davidsen 2001).

$$\mathbf{A} = \begin{bmatrix} 0 & a_{12} & 0 & 0 & a_{15} \\ a_{21} & 0 & a_{23} & 0 & 0 \\ 0 & 0 & 0 & a_{34} & a_{35} \\ a_{41} & a_{42} & 0 & 0 & 0 \\ a_{51} & a_{52} & 0 & a_{54} & 0 \end{bmatrix}$$

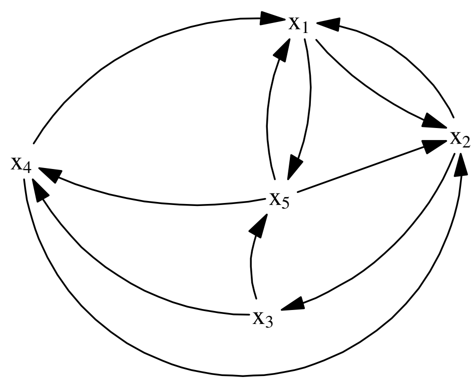

Figure 1. Causal links in a linearized model.

The causal links that have large elasticities are particularly important. If a small number of links have distinctly larger magnitudes than others, this means they define a dominant subset of model structure. Most of the time, these distinct links happen to form feedback loops in the model (Forrester 1982).

Generic model structure

A second-order model has to have a minimum of three feedback loops in order to connect the state variables with each other and their rates (Figure 2). Consequently, there are six different loop polarity combinations. Therefore, six linear homogeneous second-order models, one for each loop polarity combination, are used in the study. The generic causal loop diagram of the models, where stocks and flows are represented explicitly, is given in Figure 2.

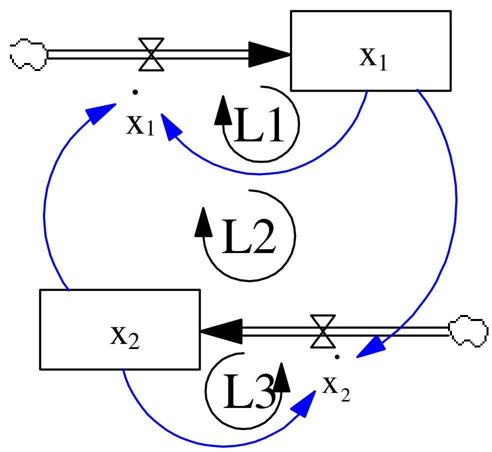

Figure 2. The 'generic' casual loop diagram of the models used in the study.

The state space representation of this system is

$$\dot{x}_1(t) = a_{11} * x_1(t) + a_{12} * x_2(t)$$
$$\dot{x}_2(t) = a_{21} * x_1(t) + a_{22} * x_2(t)$$

The matrix notation of the same system is

$$\dot{\mathbf{X}}(t) = \mathbf{A}\mathbf{X}(t)$$

where the gain matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

Experimental assessment

The loop polarities of the models, the selected gain matrices and eigenvalues are given in Table 1. The gain matrices of the models are determined subjectively. They, except the last two, which have complex conjugate eigenvalues, are set up in such a way that each one of the models has one positive and one negative eigenvalue. This means that these models have the potential to exhibit both (transient) goal-seeking behavior (i.e. the negative eigenvalue may be dominant initially) and exponential growth. The reason for such a setting is to be able to analyze the elasticities with respect to system link gains of both positive and negative eigenvalues within the same model.

The elasticities with respect to each loop gain for each model are given in Tables 2-3. The elasticities are found for a one percent change in the magnitude of the gain links. The first four models, listed in Table 2, are different than the latter two, listed in Table 3, in the sense that the latter models generate oscillations. First, the four models in Table 2 are discussed.

Table 1. The gain matrices of the models used, their loop polarities and eigenvalues.

Loop Polarities			**Gain Matrix A**	**Eigenvalues**	
L1	**L2**	**L3**		λ_1	λ_2
+	+	+	$\mathbf{A}=\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$	3	-1
–	+	–	$\mathbf{A}=\begin{bmatrix} -0.7 & 0.3 \\ 0.5 & -0.1 \end{bmatrix}$	0.089	-0.89
–	+	+	$\mathbf{A}=\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$	1.41	-1.41
+	–	–	$\mathbf{A}=\begin{bmatrix} 0.2 & 0.1 \\ -1 & -1 \end{bmatrix}$	0.11	-0.91
–	–	–	$\mathbf{A}=\begin{bmatrix} -2 & -2 \\ 2 & -2 \end{bmatrix}$	-2+2i	-2-2i
+	–	+	$\mathbf{A}=\begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$	2+2i	2-2i

One might expect that a positive (negative) loop would contribute to positive (negative) eigenvalue (i.e. exponential growth (decline)) and hamper the behavior associated with the negative (positive) eigenvalue (i.e. exponential decline (growth)). However, upon examining Table 2, one observes that when the central loop (L2) has positive polarity the elasticities of both eigenvalues with respect to that loop are positive. Thus its contribution is positive for both behavior modes represented by the two eigenvalues. On the other hand, the elasticity (i.e. the contribution) of central negative loop is negative for either eigenvalue. The second observation is that although the minor loops conform to the traditional expectation stated above when the central loop is positive, they do not when the central loop is negative, which is the case in the Model 4. There the elasticities of both eigenvalues are positive with respect to both positive and negative minor loop gains.

In the Model 1, the contribution of the central loop is higher for the negative eigenvalue than that for the positive one (0.9975 vs. 0.3325). Looking at the elasticities of the negative eigenvalue of the model it seems that the central positive loop tries to bring the system to equilibrium while the minor positive loops resist it (Table 2).

For the Model 2, as the minor negative loops put up a greater ‘resistance’ to the positive eigenvalue (its elasticities are –1.500 and –0.8966 with respect to the minor loops) than their ‘support’ for the negative one (its elasticities are 0.6350 and 0.02180 with respect to the minor loops), the contribution of the central positive loop for the positive eigenvalue is also higher than that for the negative one (1.7003 vs. 0.1718) (Table 2).

Table 2. The elasticities of eigenvalues with respect to each loop gain for the first four models.

	Model 1 (+ + +)[†]		**Model 2** (− + −)		**Model 3** (− + +)		**Model 4** (+ − −)	
Loops \ **Eigenvalues**	λ_1 (+)[‡]	λ_2 (–)	λ_1 (+)	λ_2 (–)	λ_1 (+)	λ_2 (–)	λ_1 (+)	λ_2 (–)
L1 (minor)	0.1669	-0.4994	-1.500	0.6350	-0.1032	0.6039	1.980	0.01938
L2 (central)	0.3325	0.9975	1.7003	0.1718	0.2497	0.2497	-0.8932	-0.1079
L3 (minor)	0.1669	-0.4994	-0.8966	0.02180	0.6039	-0.1032	0.7953	1.195

[†] The notations in parentheses denote the polarities of the central loop and the two minor loops.

[‡] λ_1 is the (+) eigenvalue and λ_2 is the (–) eigenvalue throughout the text.

In the Model 3, however, the elasticies suggest that the minor loops are the main driving force behind the model's both transient and steady-state behavior modes with the elasticity of the negative (positive) eigenvalue with respect to the central positive loop, 0.2497 being less than that with respect to the negative (positive) minor loop, 0.6039 (Table 2).

The interesting phenomenon that is common in the first three models is that the central loop, which has positive polarity, takes part in the unfolding of the system behavior whether it is exponential growth or decay. On the other hand, the minor loops become dominant for exponential growth if their polarity is positive and for exponential decay if their polarity is negative, in line with their conventional definitions.

The Model 4 is the only one amongst the six models with a central negative loop that does not generate cyclic behavior. Because of this particular feature, one observes different mechanisms at work between the model's structure and its behavior. Now, both negative and positive eigenvalues have negative elasticity values with respect to the central loop gain (-0.8932 for λ_1 and –0.1079 for λ_2). This means the central loop whose polarity is negative resists both potential behavior modes (i.e. exponential growth and exponential decay). On the other hand, the elasticities of eigenvalues are positive with respect to both positive and negative minor loops. In addition, the elasticities of the eigenvalues are larger with respect to the minor loops that have the same sign with them. Thus the minor negative loop is dominant and the minor positive loop plays a secondary role when the system exhibits exponential decay. Similarly, the minor positive loop is dominant and the minor negative loop plays a secondary role when the dominant behavior is exponential growth (Table 2). In short, the minor loops, regardless of their polarities drive (i.e. has positive elasiticity for) both behavior modes. This model is particularly interesting because neither the positive nor the negative loops completely comply with the roles typically expected from them.

The last two models have complex conjugate eigenvalues and generate cyclic behavior. The real component of the complex eigenvalue corresponds to the fractional expansion (growth) or contraction (decay) of the envelope within which the cyclic behavior unfolds. The imaginary component corresponds to the frequency of the oscillations. The central loops of both models have negative polarity. The models are symmetric around their central loops (Table 1).

Table 3. The elasticities of eigenvalues with respect to each loop gain for the last two models.

	Model 5 (− **—** −)[†]		**Model 6** (+ **—** +)	
Loops \ Eigenvalues	**Real(λ)**	**Im(λ)**	**Real(λ)**	**Im(λ)**
L1 (minor)	0.5	-0.0006	0.5	-0.0006
L2 (central)	0	0.499	0	0.499
L3 (minor)	0.5	-0.0006	0.5	-0.0006

[†] The notations in parentheses denote the polarities of the central loop and the two minor loops.

In both models, the contribution of the central loop is zero for the real part (which corresponds to the exponential envelope) and positive for the imaginary part (which corresponds to the frequency of the oscillations). On the other hand, the contributions of the minor loops are positive for the real part and negative for the imaginary part (Table 3). In addition, the magnitude of the contribution of the central loop for the imaginary part is much higher than those of the minor loops. Thus the minor loops are responsible for the exponential envelope whereas the central loop is in charge of the generation of the oscillations. In brief, although the polarity of the central loop is negative it does not take any part in the contraction or expansion of the envelope as it might be expected if one follows the traditional definition of negative loops. It merely serves as a means to propagate the disturbance from one minor loop to the other thus generating the oscillations. Nevertheless, the minor loops drive the contraction in the Model 5 and expansion in the Model 6 echoing their conventional definitions.

One might suspect that these findings may be special cases due to the coincidental selection of gain magnitudes. It may also be the case that not all possible structure-behavior relations are covered in the experimental phase. In order to make sure that nothing is left out in the empirical analysis and establish solid results, the above introductory discussion is reinforced with an analytic proof in the following section.

Analytical assessment

The eigenvalues of a linear model are found by solving the following equation:

$$\left|\mathbf{A}-\lambda\mathbf{I}\right|=0$$

where $|\bullet|$ denotes the determinant of $\bullet$, $\mathbf{A}$ is the gain matrix, λ are the eigenvalues, and $\mathbf{I}$ is the identity matrix.

Solving the above equality gives the eigenvalues of the matrix **A**:

$$\begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{bmatrix} = \lambda^2 - (a_{11}+a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

$$\lambda_1 = \frac{(a_{11}+a_{22}) + \sqrt{(a_{11}+a_{22})^2 - 4(a_{11}a_{22}-a_{12}a_{21})}}{2} = \frac{(a_{11}+a_{22}) + \sqrt{(a_{11}-a_{22})^2 + 4a_{12}a_{21}}}{2} \quad (2a)$$

$$\lambda_2 = \frac{(a_{11}+a_{22}) - \sqrt{(a_{11}+a_{22})^2 - 4(a_{11}a_{22}-a_{12}a_{21})}}{2} = \frac{(a_{11}+a_{22}) - \sqrt{(a_{11}-a_{22})^2 + 4a_{12}a_{21}}}{2}. \quad (2b)$$

In order to show how eigenvalues change as a result of a small change in any gain it suffices to look into the partial derivatives of both eigenvalues with respect to each gain. Recall that the partial derivatives measure the sensitivity values of eigenvalues to the entries of gain matrix and that elasticity is just the normalized expression of sensitivity. Thus, they essentially give the same information. The analysis of these derivatives reveals the effect of feedback loops on model behavior.

First, the sensitivities of eigenvalues to minor loop gains are analyzed. The partial derivatives of eigenvalues λ_1 and λ_2 with respect to a_{11}:

$$\frac{\partial \lambda_1}{\partial a_{11}} = \frac{1}{2}\left(1 + \frac{(a_{11}-a_{22})}{\sqrt{(a_{11}-a_{22})^2 + 4a_{12}a_{21}}}\right) \quad (3a)$$

and similarly,

$$\frac{\partial \lambda_2}{\partial a_{11}} = \frac{1}{2}\left(1 - \frac{(a_{11}-a_{22})}{\sqrt{(a_{11}-a_{22})^2 + 4a_{12}a_{21}}}\right). \quad (3b)$$

The partial derivatives of the same eigenvalues with respect to a_{22}:

$$\frac{\partial \lambda_1}{\partial a_{22}} = \frac{1}{2}\left(1 + \frac{(a_{22}-a_{11})}{\sqrt{(a_{22}-a_{11})^2 + 4a_{12}a_{21}}}\right) \quad (4a)$$

and

$$\frac{\partial \lambda_2}{\partial a_{22}} = \frac{1}{2}\left(1 - \frac{(a_{22}-a_{11})}{\sqrt{(a_{22}-a_{11})^2 + 4a_{12}a_{21}}}\right). \quad (4b)$$

Let's denote $\zeta_1 = (a_{11} - a_{22}) \Big/ \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}$ in Eq. 3 and $\zeta_2 = (a_{22} - a_{11}) \Big/ \sqrt{(a_{22} - a_{11})^2 + 4a_{12}a_{21}}$ in Eq. 4. For real eigenvalues, then the decomposition of the total change in the magnitude of Eq. 2 is as follows: the magnitude of change in $(a_{11} + a_{22})$ term is 1/2 and those in $\sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}$ term are $\zeta_1/2$ and $\zeta_2/2$ due to a unit change in a_{11} or a_{22}, respectively (Eq. 2). Looking at Eq. 3-4, the magnitudes of both eigenvalues increase as long as ζ_1 and ζ_2 are larger than one. This is the case when the polarity of central loop is negative (i.e. $4a_{12}a_{21} < 0$). In other words, a change in the magnitudes of one (or both) of the minor loop gains, a_{11} and a_{22} causes the magnitudes of both eigenvalues to change in the same direction. If the polarity of central loop is positive ($4a_{12}a_{21} > 0$), then ζ_1 and ζ_2 are less than one. Thus a change in the magnitudes of one (or both) of the minor loop gains, a_{11} and a_{22} causes the magnitudes of the positive and negative eigenvalues to change in the same and opposite directions, respectively. If $4a_{12}a_{21} = 0$ then the net change is zero but in this case, there is no central loop anyway.

Now, the sensitivities to the link gains a_{12} and a_{21} are analyzed. The partial derivative of λ_1 with respect to a_{12} is,

$$\frac{\partial \lambda_1}{\partial a_{12}} = \frac{1}{2}\left(0 + \frac{1}{2} \frac{4a_{21}}{\sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}} \right)$$

$$\frac{\partial \lambda_1}{\partial a_{12}} = \frac{a_{21}}{\sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}} \tag{5a}$$

and of λ_2,

$$\frac{\partial \lambda_2}{\partial a_{12}} = -\frac{a_{21}}{\sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}}. \tag{5b}$$

Finally the partial derivatives of both eigenvalues with respect to a_{21} are,

$$\frac{\partial \lambda_1}{\partial a_{21}} = \frac{a_{12}}{\sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}} \tag{6a}$$

and

$$\frac{\partial \lambda_2}{\partial a_{21}} = -\frac{a_{12}}{\sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}}. \tag{6b}$$

The partial derivatives have the same sign as their corresponding eigenvalues as long as the links a_{12} and a_{21} are of the same polarity. In other words, a change in the magnitudes of one (or both) of the central loop gains, a_{12} and a_{21} causes the magnitudes of both eigenvalues to change in the same direction if the polarity of the central loop is positive.

For complex eigenvalues, the partial derivatives derived above can be used with some modification (Eq. 4-5). The only distinction now is that the real and imaginary parts of the eigenvalues need to be treated separately. Hence, the sensitivity values take the following form:

$$\frac{\partial \mathrm{Real}(\lambda)}{\partial a_{11}} = \frac{\partial \mathrm{Real}(\bar{\lambda})}{\partial a_{11}} = \frac{1}{2}, \qquad \frac{\partial \mathrm{Real}(\lambda)}{\partial a_{22}} = \frac{\partial \mathrm{Real}(\bar{\lambda})}{\partial a_{22}} = \frac{1}{2} \tag{7a}$$

$$\frac{\partial \mathrm{Im}(\lambda)}{\partial a_{11}} = \frac{1}{2}\left(\frac{(a_{11}-a_{22})}{\sqrt{(a_{11}-a_{22})^2 + 4a_{12}a_{21}}} \right), \qquad \frac{\partial \mathrm{Im}(\lambda)}{\partial a_{22}} = \frac{1}{2}\left(\frac{(a_{22}-a_{11})}{\sqrt{(a_{22}-a_{11})^2 + 4a_{12}a_{21}}} \right) \tag{7b}$$

$$\frac{\partial \mathrm{Im}(\bar{\lambda})}{\partial a_{11}} = \frac{1}{2}\left(-\frac{(a_{11}-a_{22})}{\sqrt{(a_{11}-a_{22})^2 + 4a_{12}a_{21}}} \right), \qquad \frac{\partial \mathrm{Im}(\bar{\lambda})}{\partial a_{22}} = \frac{1}{2}\left(-\frac{(a_{22}-a_{11})}{\sqrt{(a_{22}-a_{11})^2 + 4a_{12}a_{21}}} \right) \tag{7c}$$

The expressions in Eq. 7b-c, which correspond to sensitivity values, are complex numbers with zero real parts. The gains a_{12} and a_{21}, on the other hand, have no effect on the real part and the magnitude of change in the imaginary part is as given in the right-hand side of Eq. 5-6. Note that those expressions are also complex numbers with zero real parts.

In the following analysis, the partial derivatives derived above are used to show how eigenvalues respond to changes in the gain matrix of each of the six loop configurations.

Models with central positive loop

Model 1 (+ + +)

Let's assume the link gains are such that $a_{12}a_{21} > a_{11}a_{22}$ so that there is one (–) and one (+) eigenvalue of the model (Eq. 2). Then,

1. A change in a_{11} and/or a_{22} (as long as $a_{12}a_{21} > a_{11}a_{22}$):
Since all gains are positive, $a_{21}a_{12} > 0$ and hence $\zeta_1 < 1$ and/or $\zeta_2 < 1$ (Eq. 3-4). Then the magnitude of the positive eigenvalue changes in the same direction whereas that of the negative one changes in the opposite direction. Therefore, the minor positive loops' contribution is positive for the positive eigenvalue and negative for the negative eigenvalue.

2. A change in a_{12} and/or a_{21} changes the magnitudes of both eigenvalues in the same direction (Eq. 5-6). Therefore, the central positive loop's contribution is positive for both eigenvalues!

Model 2 ($-+-$)

Let's assume the link gains are such that $a_{12}a_{21} > a_{11}a_{22}$ so that there is one (–) and one (+) eigenvalue of the model. Then,

1. A change in $|a_{11}|$ and/or $|a_{22}|$ (as long as $a_{12}a_{21} > a_{11}a_{22}$):

Since the signs of a_{12} and a_{21} are the same, $a_{12}a_{21} > 0$; hence $\zeta_1 < 1$ and/or $\zeta_2 < 1$ (Eq. 3-4). Then the magnitude of the positive eigenvalue changes in the opposite direction whereas that of the negative one changes in the same direction. Therefore, the minor negative loops' contributions are negative for the positive eigenvalue and positive for the negative eigenvalue.

2. A change in a_{12} and/or a_{21} changes the magnitudes of both eigenvalues in the same direction (Eq. 5-6). Therefore, the central positive loop's contribution is positive for both eigenvalues!

Model 3 ($-++$)

1. A change in a_{11} assuming $a_{11} > 0,\ a_{22} < 0$:

The polarity of the central loop is positive, thus the signs of a_{12} and a_{21} are the same and $a_{12}a_{21} > 0$; hence $\zeta_1 < 1$ (Eq. 3). Then the magnitude of the positive eigenvalue changes in the same direction whereas that of the negative one changes in the opposite direction. Therefore, the minor positive loops' contribution is positive for the positive eigenvalue and negative for the negative eigenvalue.

2. A change in $|a_{22}|$ assuming $a_{11} > 0,\ a_{22} < 0$:

The magnitude of the positive eigenvalue changes in the opposite direction whereas that of the negative eigenvalue changes in the same direction (Eq. 4). Therefore, the minor negative loop's contribution is negative for the positive eigenvalue and positive for the negative eigenvalue.

3. A change in a_{12} and/or a_{21} changes the magnitudes of both eigenvalues in the same direction (Eq. 5-6). Therefore, the central positive loop's contribution is positive for both eigenvalues!

Models with central negative loop

It turns out that such models are capable of exhibiting either cyclic or non-cyclic behavior depending on the relative magnitudes of the loop gains (Eq. 2). When

$(a_{11}-a_{22})^2 \geq |4a_{12}a_{21}|$ the models do not exhibit cyclic behavior; when $(a_{11}-a_{22})^2 < |4a_{12}a_{21}|$ they do. The two cases are analyzed below.

$(a_{11} - a_{22})^2 \geq |4a_{12}a_{21}|$:

The models have two real eigenvalues and do not exhibit cyclic behavior (Eq. 2):

Model 4 ($+ - -$)

Let's assume the link gains are such that $|a_{11}a_{22}| > |a_{21}a_{12}|$ so that there is one (–) and one (+) real eigenvalue of the model.

1. A change in a_{11} assuming $a_{11} > 0,\ a_{22} < 0$:

Since the polarity of the central loop in this model is negative, a_{12} and a_{21} are of the opposite signs, $a_{21}a_{12} < 0$ and hence $\zeta_1 > 1$ (Eq. 3). Then the magnitudes of both eigenvalues change in the same direction. Therefore, the minor positive loop's contribution is positive for both eigenvalues!

2. A change in $|a_{22}|$ assuming $a_{11} > 0,\ a_{22} < 0$ yields a similar result (Eq. 4). That is to say, the magnitudes of both eigenvalues change in the same direction. Therefore, the minor negative loop's contribution is positive for both eigenvalues also!

3. A change in $|a_{12}|$ and/or $|a_{21}|$ (as long as $|a_{11}a_{22}| > |a_{12}a_{21}|$) leads to a change in the magnitudes of both eigenvalues in the opposite direction (Eq. 5-6). Therefore, the central negative loop's contribution is negative for both eigenvalues.

Model 5 ($- - -$) and Model 6 ($+ - +$)

Model 5 and Model 6 have two real eigenvalues both of which are always either negative or positive, respectively (Eq. 2).

1. A change in $|a_{11}|$ assuming $|a_{11}| > |a_{22}|$:

Since the polarity of the central loop in this model is negative, a_{12} and a_{21} are of the opposite signs, $a_{21}a_{12} < 0$ and hence $\zeta_1 > 1$ (Eq. 3). Then, the magnitude of the larger eigenvalue changes in the same direction while that of the smaller one changes in the opposite direction.

2. A change in $|a_{22}|$ assuming $a_{11} > 0,\ a_{22} < 0$ yields the opposite result. That is to say, the magnitude of the larger eigenvalue changes in the opposite direction while that of the smaller one changes in the same direction (Eq. 3).

3. A change in $|a_{12}|$ and/or $|a_{21}|$ (as long as $|(a_{11}-a_{22})^2|>|4a_{12}a_{21}|$) leads to a change in the magnitude of the both eigenvalues of the system in the opposite direction (Eq. 5-6). Therefore, the central negative loop's contribution is negative for both eigenvalues.

$(a_{11}$ - $a_{22})^2 < |4a_{12}a_{21}|$:

The models have complex conjugate eigenvalues and exhibit cyclic behavior (Eq. 2). Of the three models with a central negative loop, the real part of the eigenvalues of the Model 5 and the Model 6 are negative and positive, respectively. Therefore, the first exhibits converging oscillations while the other exhibits diverging oscillations. Model 4, on the other hand, can show either diverging or converging oscillations depending upon whether the gain of the positive or negative minor loop is greater than the other, respectively. If the magnitudes of the gains are equal, however, it exhibits neutral stability. Then the slightest perturbation in either gain causes the system to shift to diverging or converging oscillations.

Model 4 ($+--$)

Let a_{11} and a_{22} be the gains of the minor positive and negative loops, respectively.

1. A change in $|a_{11}|$ and/or $|a_{22}|$:

a. $|a_{11}|>|a_{22}|$:

In this case, the real part is positive (Eq. 2). Thus, the behavior mode of the system is diverging oscillations. A change in $|a_{11}|$ changes the magnitude of the real part of the complex eigenvalue pairs in the same direction (Eq. 7a). However, a change in a_{22} changes the magnitude of the real part in the opposite direction. This case then represents what typically could be expected from positive and negative loops.

b. $|a_{11}|=|a_{22}|$:

This corresponds to the case where there is neutral stability (i.e. limit cycles). Hence, a slightest change in either gain makes the real part non-zero. If there is an increase (decrease) in $|a_{11}|$ the real part becomes positive (negative) and the exponential envelope diverges (converges). The opposite is true for a change in $|a_{22}|$.

c. $|a_{11}|<|a_{22}|$:

In this case, the real part is negative (Eq. 2). Thus, the behavior mode of the system is converging oscillations. A change in $|a_{22}|$ changes the magnitude of the real part of the complex eigenvalue pairs in the same direction (Eq. 7a). However, a change in $|a_{11}|$ changes the magnitude of the real part in the opposite direction. This is again what typically could be expected from positive and negative loops.

The change in either gain, however, in all three cases causes a change on the imaginary part in the opposite direction (Eq. 7b-c). This means an increase (decrease) in the magnitude of either gain decreases (increases) the frequency of oscillations.

2. A change in $|a_{12}|$ and/or $|a_{21}|$:
A change in $|a_{12}|$ and/or $|a_{21}|$ (in other words a change in the central loop gain) only affects the imaginary part. In other words, these links play no part in dynamics associated with the real part of the eigenvalue (Eq. 7a). A change in the magnitude of either link gain results in a corresponding change in the same direction in the magnitude of the imaginary part (Eq. 7b-c). Thus these links, which form the central loop, only govern the frequency of oscillations. This is different than the previous case in which the minor loop gains affect both the real and imaginary parts of the eigenvalue.

Model 5 ($- - -$) and Model 6 ($+ - +$)

1. A change in a_{11} and/or a_{22}:
Such a change causes the magnitude of the real part of the complex eigenvalue pairs to change in the same direction (Eq. 7a). Therefore, the exponential envelope around the oscillations gets larger or smaller. The effect on the imaginary part (i.e. on the frequency of oscillations), however, is not straightforward. Two different cases need to be taken into account as follows:

a. $|a_{11}| > |a_{22}|$*:*
The sensitivity value is negative with respect to the gain with larger magnitude (in this case, it is a_{11}) but positive with respect to the other (Eq. 7b-c). Moreover, the magnitude of the elasticity is always larger for the gain with larger magnitude. This is interesting because having the same polarity one thinks the effects of the minor loop gains on the imaginary part would be the same. However, in fact, increasing (decreasing) the magnitude of the minor loop with the smaller (larger) gain makes the system more 'symmetric' causing the oscillation potential to increase. The case where $|a_{11}| < |a_{22}|$ is similar.

b. $|a_{11}| = |a_{22}|$*:*
Both the signs (which is negative) and the magnitudes of the sensitivities are the same for both gains. This seems to constitute a special case because of the symmetry of the system. In this case, the slightest change in the gains of the minor loops make the system less symmetric and hence, less suitable for the propagation of disturbances from one subsystem to the other. The models whose matrices are given in Table 3 are of this case.

2. A change in $|a_{12}|$ and/or $|a_{21}|$:
This case is exactly the same as the one for the Model 4 above.

The analysis of the systems with complex conjugate eigenvalues sheds more light on the resistance of the central negative loop to both behavior modes when those systems have two real eigenvalues, one positive and one negative. The function of the central negative loop is to propagate disturbances between the two sub-systems represented by the minor loops. This propagation of disturbances is what creates the potential for cyclic behavior. However, if the gains of model loops are such that $(a_{11}-a_{22})^2 \geq |4a_{12}a_{21}|$ the central negative loop cannot create oscillations. What it does, however, is to restrain the two behavior modes of the system. The condition to have cyclic behavior for a second-order system with a central negative loop is then $(a_{11}-a_{22})^2 < |4a_{12}a_{21}|$.

The analytical approach reveals aspects of structure-behavior relation that are missed in the experimental phase. Moreover, it provides a solid understanding on the relation between the structure and behavior, which may not be evident at all in an empirical analysis.

The principle on the significance of the loop arrangement on behavior

The analysis revealed that the relative location and magnitude of loops in the system structure also has a say in the role played by a feedback loop in the unfolding of system behavior. This fact is formulated in a principle for second-order linear models as follows:

The Principle: For a second-order linear system with the potential of exhibiting both exponential growth and exponential decay (i.e. having real eigenvalues, one being positive and the other negative) the central loop, if its polarity is positive, contributes to the unfolding of both behavior modes. However, it works against both behavior modes if its polarity is negative. The minor loops function in accordance with the typical definitions attributed to loops based on their polarities in the generation of the two behavior modes if the polarity of the central loop is positive. In contrast, minor loops are responsible for the unfolding of both behavior modes regardless of their polarities if the polarity of the central loop is negative. For the system to exhibit cyclic behavior (i.e. to have complex conjugate eigenvalues) it has to have a negative central loop. Moreover, the magnitudes of the loop gains have to be more or less comparable if the polarities of the minor loops are the same or the magnitude of the central loop gain need to be considerably larger than the magnitudes of the minor loop gains if minor loops are of opposite polarities. The condition to have cyclic behavior for a second-order linear system with a central negative loop is then $(a_{11}-a_{22})^2 < |4a_{12}a_{21}|$. In this case, the central negative loop determines only the frequency of oscillations while the positive (negative) minor loops are responsible for the expansion (contraction) of the envelope around the oscillations.

Final Remarks

Typically, negative loops are associated with goal-seeking behavior while positive loops are thought to be responsible for exponential growth. On the other hand, this study shows that the relative location and magnitude of loops with respect to each other in the system structure also has a say in the role played by a feedback loop in the unfolding of system behavior. The recognition of the downsides of predicting behavior from loop polarity alone is not new (Richardson 1986). The additional insight in this study comes from the recognition of, using an analytical approach, how different arrangements of the loops create different interactions between the system structure and behavior.

This fact is formulated in a principle for second-order linear models but may also have significant implications for more complex systems. Such principles bring about the opportunity for the reader "to become more aware of his or her own half-conscious rules of thumb that relate structure to behavior, to the extent that they can be explicated as principles" (Graham 1977).

The study also highlights the pitfalls of purely empirical analysis when it comes to extracting insights on the relation between structure and behavior. The analytical approach provides information on the full spectrum of structure-behavior relations while the use of empirical approach serves as a preliminary analysis tool. Nonetheless, empirical analysis is still the most efficient way to analyze highly complex dynamic models.

It is hoped that the findings in this paper will inspire more studies on this topic. For example, an immediate extension of this study would be carrying out a similar analysis on higher order (e.g. third order) linear models, which are a great deal more complex than second order ones. Another research direction is the application of analogous analyses on low-order nonlinear models and extracting similar principles regarding the structure-behavior associations.

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