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A Model of Sustainable Development with Uneven Growth of Labor Force

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Abstract

This paper elaborates the notion of viable quasi-periodic motion bounded in the phase space that generalizes stationary growth and stationary cyclical growth. This elaboration has been supported by the simulation experiments, based on the original hypothetical law (HL) of capital accumulation, and by statistical data. The long wave is exposed not as long-term fluctuations around an equilibrium trend but as a quasi-periodic non-equilibrium trend and stochastic attractor. This presentation differs essentially from the neo-classical view on economic growth as a convergence towards equilibrium. The fundamental equation of neo-classical growth is a special case of the more general dynamic regularity, presented as a direct consequence of the HL.

The application of the HL with exogenous growth of labor force to the U.S. economy has shown that the moderation of the secular tendency of the average profit rate to fall is conditioned by the society’s strategy to invest in natural capital.

Key words: capital accumulation, sustainable development, Kalman filtering, stochastic attractor

> *… the threat to corporations, and indeed to other human institutions, arises from the possibility of social and economic breakdown. The internal threat is much more serious than external military threat.*
>
> Jay W. Forrester[1]

Introduction

Almost all of the empirical work on economic growth takes place in a neo–classical framework. “…the literature is essentially concerned with trying to identify empirically the *ex post* contributions of a range of factors to the observed rate of growth. In this view of the world, the role of profits in capitalism is effectively non–existent” (Ormerod 1996: 2). There is a growing dissatisfaction with these enormous intellectual efforts expended with a very few clear conclusions.

The book (Ryzhenkov 2000a) and subsequent papers (Ryzhenkov 2000b, 2001, 2002) have defined and refined *the hypothetical law of advancing capitalism* (HL) as a system of non–linear ordinary differential equations. The state variables are the relative wage,

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[1] Forrester 1993: 1-6.14-1-6.15.

employment ratio, unit gross rent, man-made capital–output ratio, natural capital–output ratio, indicated natural capital–output ratio and unit depreciation of the natural capital.

The HL, presented as a system dynamics model in the intensive form, reflects the dialectical interaction between factors that tend to lower the average rate of profit and those that counteract this tendency. Conversion of profit into capital and sustained expansion for a number of years eventually results in a tight labor market, rising real wages, and an acceleration of capital–labor substitution. As this process tends to raise the capital–labor ratio, it also tends to lower the average rate of profit. When this tendency outweighs the counteracting tendencies, a recession follows the expansion.

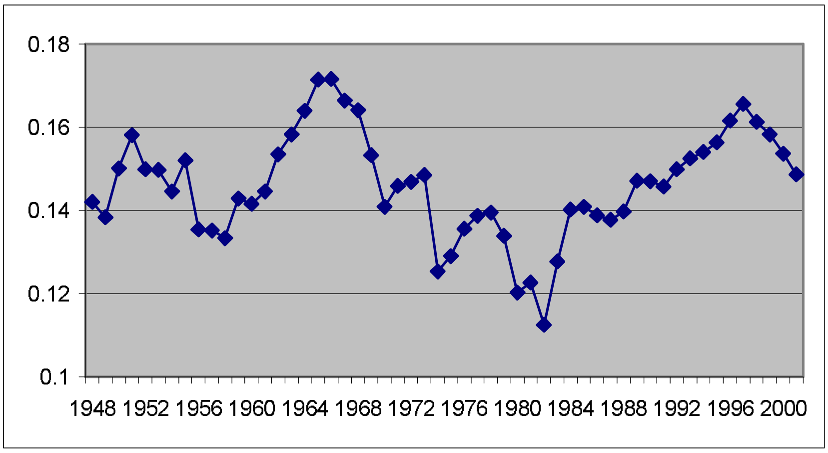

Figure 1 The average profit rate in the USA, 1948-2001

After the Second World War, the American economy passed peaks of the Kondratiev cycles twice: 1966-1969, 1997-2000 (Figure 1).[2] The HL explains not only the long-term quasi-periodic fluctuations of the average profit rate. It sheds light on a secular tendency of the average profit rate to fall that has been typical for the U.S. economy at least from the middle 1960-s (Ryzhenkov 2002a, 2002b).

In this original model, knowledge generates economic growth through technological progress, including creative innovation favorable for employment. An induced technological progress can facilitate together the employment ratio, profitability of both man-made and natural capital if the requirements outlined are satisfied.

The present downturn in the long wave is not only a regularly recurrent phase of the long wave. Its additional pains are characteristic of childbirth of the natural capitalism. The 'old' industrial capitalism is experiencing a dialectical negation, or creative

[2] See Ryzhenkov 2002b.

destruction. The system dynamics approach could be helpful for shortening and lessening disorder and distress of this major global transformation.[3]

The HL has been tested against facts and has undergone numerous laboratory experiments. The non–linear feedback relationships, measurement errors violate the maintained hypotheses of most single–equation econometric techniques. However, the Powell hill climbing algorithm and Kalman filtering offer a promising approach to formal estimation of the HL (Ryzhenkov 2001, 2002a). The simulation software VENSIM developed by Ventana Systems, Inc. allows applying these techniques.

On the one hand, this paper shows that both the neo-classical theory and post-marxian theory, exposed below, agree on a fundamental positive role of a rate of growth of labor force for an economic growth rate and average profit rate. On the other hand, this paper reveals advantage of a post-marxian disequilibrium approach to the modern capitalist economy over the equilibrium approach of the neo-classical school.

The HL, especially in its probabilistic form, enables, in particular, to generalize the fundamental equation of neoclassical economic growth and to demonstrate that fragile stationary economic growth is not practically feasible or realizable, unlike the neo-classical contention.

The given formulation of the HL is not final. It requires further refinement. One of the HL assumptions postulates that the labor force is constant or changes exponentially over time. This paper reports on procedures required for substituting this assumption by a hypothesis provided by the Bureau of Labor Statistics of the U.S. Department of Labor. This substitution and computer simulations help to redefine the previous recommendations on resilient investment policies for achieving strongly sustainable development in the USA in the XXI century given in the previous papers (Ryzhenkov 2001, 2002a, 2002b). This paper demonstrates, in particular, that newly suggested policies enable to contain quasi-periodic dynamics in necessary bounds thus preventing the potential class conflicts over distribution of income from escalating.

1 Generalizing the fundamental equation of neo-classical economic growth

The fundamental equation of neo-classical economic growth (FENEG) corresponds to the equation (6) of Solow's paper (1956). Explicitly accounting for unemployment, resource rent and natural capital enables to generalize this equation in the *hypothetical law* (HL).

1.1 An exogenous labor force growth in Solow's model

The Solow (1956) paper starts with pointing out that in the Harrod – Domar model (HDM) even for the long run the economic system is at best on a knife–edge of equilibrium growth. Were the magnitudes of key parameters — the saving ratio, the capital–output ratio, the rate of increase of labor force — to slip ever so slightly from dead center, the consequence would be either growing unemployment or prolonged inflation.

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[3] "Because it [natural capitalism] is both necessary and profitable, it will subsume traditional industrialism within a new economy and new paradigm of production, just as industrialism previously subsumed agrarianism" (Lovins et al. 1999: 158).

The paper argues that this fundamental opposition of warranted and natural rates turns out in the end to follow from the critical assumption that production takes place under conditions of fixed proportions. There is no possibility of substituting labor for capital in production in the HDM. If this assumption is abandoned, the knife–edge notion of unstable balance seems to go with it.

A closed economy produces net output designated by P. Part of it is consumed and the rest, qP, is saved and invested without material delay. The national stock of capital K takes the form of the composite commodity. Net investment and the rate of increase of this capital stock are identical (1.1). Here and below time derivatives are denoted by a dot, while a hat indicates growth rates. Two factors of production, fixed capital and labor, are used.

$$\dot{K} = qP \qquad (1.1)$$

Aggregate savings are independent of the functional distribution of income between wages and profits; savings are smoothly transformed into investment via an appropriate interest rate, quite independently of the going profit rate. The rate of labor input is L. Technological possibilities are represented by a production function (1.2). It shows constant returns to scale (homogeneity of first degree).

$$P = F(K,L). \qquad (1.2)$$

Inserting (1.2) in (1.1) we get

$$\dot{K} = qF(K,L)\,. \qquad (1.3)$$

It is assumed (1.4) that the labor force increases at a constant relative rate n as a result of exogenous population growth.

$$L(t) = L_0e^{nt}\,. \qquad (1.4)$$

Full employment of labor and capital is perpetually maintained in this model. Therefore, it is possible to insert (1.4) in (1.3) to get

$$\dot{K} = qF(K,L_0e^{nt})\,. \qquad (1.5)$$

The marginal productivity equation determines the wage rate $\frac{\partial F(K,L)}{\partial L} = w.$ A complete set includes the latter equation together with (1.3) and (1.4). A similar marginal productivity equation for capital determines the real rental per unit of time for the services of capital stock. According to Solow, once we know the time path of capital stock and that of labor force, we can compute from production function the corresponding time path of real output.

1.2 A derivation of the fundamental equation of neo–classical growth

A new variable (capital–labor ratio, or capital intensity) is introduced $r = K/L$. Hence $K = rL = rL_0e^{nt}$. After differentiating with respect to time and substituting in (1.5), we get

$$\dot{K} = L_0 e^{nt}(\dot{r} + nr) = qF(K, L_0 e^{nt}).$$

Due to constant returns to scale, it is possible to divide both variables in F by $L = L_0e^{nt}$ if F is multiplied by the same factor. Thus

$$L_0 e^{nt}(\dot{r} + nr) = qL_0 e^{nt} F(\frac{K}{L_0 e^{nt}}, 1).$$

Dividing out the common factor, we arrive finally at

$$\dot{r} = qF(r,1) - nr. \qquad (1.6)$$

The differential equation (1.6) involves the capital–labor ratio alone. The subsequent literature (Jones 1976: 75) considers (1.6) as the fundamental equation of neo–classic economic growth (FENEG).

The rate of change of the capital–labor ratio, r, is determined by the difference between the amount of saving (and investment) per worker and the amount required to keep the capital–labor ratio constant as the labor force grows. When $\dot{r} = 0$, the capital–labor ratio is a constant, and the capital must be expanded at the same rate as the labor force, namely n.

With constant returns to scale, marginal productivities depend only on the capital–output ratio r, and not on any scale quantities. The factor markets in the Solow model work perfectly since the wage rate and profit adjust smoothly and instantaneously to changing circumstances. The rate of profit, being a reflection of how scarce capital in relation to the labor force, is not important factor for the growth rate in this model.

1.3 A stationary state for neutral technological change and the Cobb–Douglas production function

Let a production function is the Cobb–Douglas function, while neutral technological change is reflected as an exponential growth factor. Then we alter (1.2) to get:

$$P = e^{\gamma t} K^{\alpha} L^{\beta},$$

where $\gamma \geq 0$, $0 < \alpha < 1$, $\beta = 1 - \alpha$. The special property of the Cobb–Douglas function is that the relative share of labor is constant at $1 - \alpha$.

A growth rate of a net national product is $\hat{P} = \gamma + \alpha\hat{K} + (1-\alpha)\hat{L} = \gamma + \alpha q/s + (1-\alpha)n$. The higher a growth rate of the labor force (n), the faster is the economic growth. In the long run, the capital stock increases at the relative rate $n + \gamma/\beta$ compared with n in the case of no technical change. The eventual rate of increase of real output is not $n + \alpha\gamma/\beta$, as given in the original text (Solow 1956: 85), but $n + \gamma/\beta$. Consequently the capital coefficient grows eventually at rate $n + \gamma/\beta - (n + \gamma/\beta) = 0$. The rate of growth, warranted by the appropriate return to capital, asymptotically equals the natural rate, unlike the conclusion in the paper (Solow 1956: 86).

The model has a unique non–trivial stationary state that is globally asymptotically stable. For this state, the magnitudes of main variables are determined:

the capital–output ratio $s_{eq} = \dfrac{q}{\dfrac{\gamma}{1-\alpha}+n}$,

the profit rate, or *real rental* = $\dfrac{\alpha}{s_{eq}} = \dfrac{\alpha(\dfrac{\gamma}{1-\alpha}+n)}{q}$,

the growth rate $\hat{P}_{eq} = q / s_{eq} = \dfrac{\gamma}{1-\alpha}+n$ (not $n + \alpha\gamma/\beta$, as stated in (Solow, 1956: 85)).

The stationary growth rates of the wage rate, capital intensity, labor productivity, are

$$\hat{w}_{eq} = \hat{r}_{eq} = \hat{a}_{eq} = \hat{K}_{eq} - \hat{L}_{eq} = \frac{qP_{eq}}{K_{eq}} - n = \frac{qe^{\gamma t}K_{eq}{}^{\alpha}L_{eq}{}^{1-\alpha}}{K_{eq}} - n = \frac{qe^{\gamma t}}{r_{eq}{}^{1-\alpha}} - n = \frac{\gamma}{1-\alpha},$$

respectively.

Finally, the stationary capital intensity is determined as r_{eq} = $(qe^{\gamma t}/(n+\gamma/(1-\alpha)))^{1/(1-\alpha)}$.

All these relationships include the growth rate of labor force (n). In a stationary state, the higher this growth rate, the lower the capital–output ratio and capital intensity, the higher the real rental, the faster economic growth. We see that the rate of growth of the labor force plays a fundamental role in this model.

The literature on endogenous growth of labor force goes back to Th. R. Malthus, A. Smith, D. Ricardo.[4] These outstanding thinkers stressed unanimously the economic importance of growth in the supply of labor, but they disagreed about an existence of a particular relationship and/or about the strength of a relationship. Still a common property of their theories is consideration of the rate of labor force growth as an increasing function of real wage.

In an upgraded neo–classical model, the real wage (w) is itself an increasing function of capital intensity (r). The rate of growth of labor force is now $n(r)$. A further hypothesis stipulates that there is a higher real wage, occurring at a higher capital intensity r_1, such that $n(r)$ is decreasing for $r > r_1$, and may fall to zero and even beyond due to the demographic transition (Solow 1999: 657–658).[5]

1.4 A generalisation of the FENEG

A paper (Ryzhenkov 2000b) has offered the following generalization. Solow's assumptions are preserved with important exceptions: the absence of technical change, the constant returns to scale, instantaneous adjustment of the real wage and clearing of the labor market, simple reproduction are neither required nor prohibited. Transformations come next from the identity $\widehat{K/L} = \hat{K} - \hat{L}$ and (1.1)

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[4] See a review *Zeit der Oekonomen* in ZEIT–Punkte, Nr. 3/1993. New growth theory makes population growth one of its hallmarks (see Barro and Sala-I-Martin (1995) for references).

[5] The work *World Dynamics* by J. W. Forrester offered a more sophisticated explanation of the growth rate of population and hence of labor force taking into account not only material standard of living and food per capita, but crowding and pollution. See Forrester 1971.

$$\dot{K/L} = \hat{K}(K/L) - \hat{L}(K/L)$$
$$= \dot{K}/L - \hat{L}(K/L)$$
$$= qP/L - \hat{L}(K/L) .$$

Let labor productivity $a = P/L$. We have finally

$$\dot{r} = qa - \hat{L}r. \qquad (1.7)$$

This equation is the generalization looked for. In particular, the FENEG (1.6) is valid for $\hat{L} = n, a = F(r,1)$. Another specific form of (1.7) is presented below.

The most important empirical fact falsifying the general equilibrium theory is the failure of labor market to clear over long periods of time (Arrow 1994). This fact finds explanation in an alternative theory presented in this paper.

2 The original model of sustainable development

The necessity of linking both components — growth and long waves — empirically as well theoretically is as an important topic. The original system dynamics model of cyclical growth includes the stocks and flows, multiple non–linear feedback processes, and other elements of dynamic complexity. This model reflects the impact of economic activities upon natural environmental conditions. These conditions, in their turn, influence the growth rates of labor productivity and capital intensity. Policies, based on a perception of resource scarcity and pollution levels, are also reflected.

2.1 The model assumptions

A capitalist economy is restricted by natural resources. Produced capital is an embodiment of knowledge and, similarly, natural capital is a stock of information. Some conversion factors are needed for aggregating information content of different constituents. Fixed assets, labor and natural assets are essentially complementary to each other and are also substitutes to some degree depending on relative price changes.

The other most important premises are such:

(1) two social classes (capitalists and workers); the State enforces the property rights, yet the cost of such an enforcement is not treated explicitly;
(2) three factors of production — labor force, man–made fixed capital, natural capital — are homogenous and non–specific;
(3) only one aggregated good is produced for consumption, investment and circulation, its price is identically one;
(4) production (supply) equals effective demand;
(5) productive capacities can be partially idle;
(6) all wages consumed, the resource rent and a part of profits saved and invested;
(7) steady growth in the labor force that is necessarily not fully employed;
(8) a growth rate of a unit real wage rises in the neighborhood of full employment;
(9) a change in capital intensity and technical progress are not separable due to a flow of invention and innovation over time;
(10) a qualification of the labor force corresponds to technological requirements.

The product–money identity and the supply–demand equivalence stated in the third and fourth assumptions do not contradict the two–fold character of labor embodied in commodities. This model mirrors the twofold nature of labor power, the unity and contradiction of its value and use–value. The creative functions of labor market as an instrument for transmitting impulses to economic change are the focal point.

The model does not describe the formation of real income of the unemployed persons. It is assumed that a part of wages and salaries covers indirectly the needs of the unemployed. The latter do not play an active role in the model economy. Social security contributions and benefits are not shown unambiguously.

The model assumes supremacy of production over final demand. This assumption abstracts from the relative independence of final demand. It is more acceptable for the long run as for the short–run: although in the shorter run aggregate demand influences output, in the very long run output dominates over demand. Capital adapts the output to the scale of production.

The model abstracts from over–production of commodities inherent in over–production of capital during certain phases of industrial cycles. The assumption (6) simplifies definitions of the investment, saving and profit rates. It may be a key to explanation of the fact that the rate of profit on capital of order of 12 or 15 per cent per annum is compatible with a rate of economic growth of two or three and half per cent per annum.

The assumption (5) reflects the existence of excessive productive capacities. It is important for interpreting an equation for a rate of change of labor productivity (below). The assumption (7) means that the labor force grows exponentially over time. This assumption is to be substituted by a more realistic hypothesis below. The assumption (6) corresponds to the immediate aim of profit–oriented capitalist production.

2.2 The equations of the original model

The model is formulated in continuous time. Time derivatives are denoted by a dot, while growth rates will be indicated by a hat. This model consists of the following equations:

$$P = K/s; \tag{2.1}$$

$$a = P/L; \tag{2.2}$$

$$u = w/a; \tag{2.3}$$

$$\hat{a} = m_1 + m_2\widehat{(K/L)} + m_3\psi(\hat{v}) + m_5\widehat{F/L}, \tag{2.4}$$

$$\psi(\hat{v}) = SIGN(\hat{v})ABS(\hat{v})\wedge j,\ m_1 \ge 0,\ 1 \ge m_2 \ge 0,\ m_3 \ge 0,\ 1 \ge m_5 \ge 0,\ 1 \ge j > 0;$$

$$\widehat{(K/L)} = n_1 + n_2 u + n_3(v - v_c) + n_5(Z/P), \tag{2.5}$$

$$n_2 \ge 0,\quad n_3 \ge 0,\quad n_5 \ge 0,\quad 1 > v_c > 0;$$

$$v = L/N; \tag{2.6}$$

$$N = N_0 e^{nt},\ n = const \ge 0,\ N_0 > 0; \tag{2.7}$$

$$\hat{w} = -g + rv + b\widehat{(K/L)} + q\widehat{F/L},\ g \ge 0,\ r > 0; \tag{2.8}$$

$$P = C + \dot{K} + Y = wL + (1 - k)M + \dot{K} + Y; \tag{2.9}$$

$$\dot{F} = Y - Z; \tag{2.10}$$

$$Z = eP,\ 0 < e < 1; \tag{2.11}$$

$$y = Y/P \ge 0; \tag{2.12}$$

$$\hat{X} = i; \tag{2.13}$$

$f = F/P;$ (2.14)

$c = X/P;$ (2.15)

$\hat{e} = \hat{P}(e_1/e - 1)$, $e \geq e_1 > 0;$ (2.16)

$\dot{K} = kM = k[(1 - w/a)P - Y] = k[(1 - u)P - Y]$, $0 < k \leq 1;$ (2.17)

$\dot{y} = (o_1(c - f) + o_2\hat{f})y$. (2.18)

Equation (2.1) postulates a technical relation between the capital stock (K) and net output (P). The variable s is called capital–output ratio. Equation (2.2) relates labor productivity (a), net output (P) and labor input or employment (L). Equation (2.3) describes the shares of labor in net output (u).

Equation (2.4) is an extended technical progress function. It includes: the rate of change of produced capital intensity, K/L, the direct scale effect, $m_3\psi(\hat{v})$, and the rate of change of natural capital intensity, F/L. $ABS(x)$ is absolute value of x that is non–negative, x^j is x raised to the j–th power, $SIGN(x)$ is a sign of x. The parameter j will be randomized in the univariate sensitivity analysis below.

Equation (2.6) outlines the rate of employment (v) as a result of the buying and selling of labor–power. Labor force grows exponentially in (2.7). In the equation (2.8), the rate of change of the wage rate (w) depends on the employment rate (v), as in the usual Phillips relation, and on the rates of change of capital intensity (K/L) and (F/L), additionally. The capital intensity (K/L) is a proxy for qualification.

In the equation (2.9), the sum of net export, final private and public consumption is $C = P[u + (1- k)(1- u - y)]$. The net formation of produced fixed capital is $\dot{K} = kM$. The gross accumulation of natural assets Y equals the gross resource rent in monetary (or information value) terms. Equations (2.9) and (2.17) show that profit ($M = (1- u - y)P$) and incremental man–made capital ($\dot{K}$) are not equal in monetary (or information value) terms if the investment share $k < 1$.

In the equation (2.10), $\dot{F}$ is a net accumulation (loss) of the natural capital (F). Z is the net environmental damage in the equation (2.11), i.e., depletion and degradation of non–produced natural assets (land, soil, landscape, eco–systems) due to economic uses above the regeneration rate.[6] The resource use or pollution has a fixed relationship to output. The linearity of this relationship constitutes a particular case ($e = const$). A non–linear relationship (2.16) was firstly introduced in (Ryzhenkov 2001).

The rate of change of capital intensity (K/L) in the equation (2.5) is a function of the relative wage (u), difference between real employment ratio and some base ('natural') magnitude ($v - v_c$), depletion/degradation of natural capital in relation to net output (Z/P). The rate of growth of capital intensity depends on the environmental damage per unit of output (an application of the principle 'a pollution prevention pays'), in particular. A high wage share and high employment ratio promote mechanization (automation).

[6] The rate of regeneration is given by a function $Q(F, Y)$, satisfying $Q(0, Y) = 0$, $\partial Q/\partial Y > 0$ (at least for F above a certain minimal level of F) in a more detailed model of sustainable development. There is a perceived social need of directing technological progress to the development of material resources with a shorter regeneration time after the epoch of the increasing aggregate regeneration time of the resource package in use (Saeed 1994: 124–130). These aspects are skipped in this paper.

The indicated natural capital, X, may remain constant, decrease or increase exponentially in the equation (2.13). In (2.12), y is the investment ratio for the natural capital. The equation (2.18) defines an investment policy that is aimed to develop the natural capital in accordance with the indicated natural capital. A combination of proportional and derivative control over the investment in natural capital is used hereby.

This model does not treat explicitly a stock of environmental assets. The natural capital–output ratios — real, f, and indicated, c, in the equations (2.14) and (2.15) — belong to the state variables of the model.

We assume that the unit depletion (degradation) of the natural capital asymptotically declines due to substitution and structural change as in (2.16) where for $\hat{P} > 0$ and $e > e_1$, $\hat{e} < 0$. The higher the rate of economic growth, the faster is the reduction of eco–intensity (or the promotion of eco–efficiency in the narrow sense). The equation (2.16) is, likely, a better approximation than e = const > 0. An approximation of a higher order can be easily implemented in the future work.

The flow variables P, C, M, Y, and Z are measured in monetary units per year, the stock variables K and F are measured in monetary units. Respectively, these variables could be measured in bits per year and bits as well. Methods of an evaluation of their informational content need a special elaboration that goes beyond the scope of this paper.

The next peculiarity of the model is that it has only implicit delays. Due to them, the model gets rid of instantaneous adjustment to an equilibrium with full employment of labor force used by the earlier neo–classical theories of economic growth. An explicit investment delay is still set aside.

Three profit rates are defined for this economy. The first is the *average* rate of return to man–made capital $(1- u - y)/s$. The second is a *general* one, it measures a ratio of the economic surplus to the total value of produced and natural capital $(1 - u - e)/(s + f)$. The third is a *gross* (biased) profit rate $(1- u)/s$ that is more easily calculated based on the statistics with incomplete data on the natural resources.

The rate of net rent is the ratio of net unit rent to natural capital – output ratio, $(y - e)/f$. The general rate of profit is a weighted average of the rate of return to man–made capital and the rate of net rent: $(1 - u - e)/(s + f) = [s/(s + f)](1- u - y)/s + [f/(s + f)](y - e)/f$.

The average rate of profit can grow because of a rise in the capital share $(1 - u - y)$, a decline in the capital–output ratio (s), or decline in the relative price of capital goods (p/p_K). The ratio p/p_K is identically one in this one–product model.

Through a transformation of $\widehat{K/L} = \hat{K} - \hat{L}$, it is easy to derive a generalization of the FENEG:

$$\dot{\overline{K/L}} = \hat{K}(K/L) - \hat{L}(K/L) = \dot{K}/L - \hat{L}(K/L) = k(1-u-y)a - \hat{L}(K/L).$$

The FENEG is a particular case of this equation for $k(1 - u - y) = const$ and $\hat{L} = \hat{N} = n$.

The original model in an intensive form has been derived in (Ryzhenkov 2001, 2002a). It consists of seven differential equations (2.19) – (2.25) that determine *a hypothetical law of capital accumulation*:

$$\dot{s} = -\frac{1}{(1-m_5)}(m_1 + (m_2 + m_5 - 1)(n_1 + n_2 u + n_3(v - v_c) + n_5 e) + m_3\psi(\hat{v}) + m_5 \hat{f})s; \qquad (2.19)$$

$$\dot{v} = \left(k\frac{1-u-y}{s} - (n_1 + n_2u + n_3(v - v_c) + n_5e) - n\right)v; \tag{2.20}$$

$$\dot{u} = (-g + rv - m_1 + (b + q - m_2 - m_5)(n_1 + n_2u + n_3(v - v_c) + n_5e) - m_3\psi(\hat{v}) + (q - m_5)(\hat{f} - \hat{s}))u; \tag{2.21}$$

$$\dot{f} = \left((1 - m_5)\frac{(y-e)}{f} - m_1 - m_2(n_1 + n_2u + n_3(v - v_c) + n_5e) - m_3\psi(\hat{v}) - (1 - m_5)(\hat{v}+n)\right)f; \tag{2.22}$$

$$\dot{c} = (i - k\frac{1-u-y}{s} + \hat{s})c\,; \tag{2.23}$$

$$\dot{y} = (o_1(c-f) + o_2\hat{f})y\,; \tag{2.24}$$

$$\dot{e} = \left(k\frac{1-u-y}{s} + m_1 + (m_2 + m_5 - 1)(n_1 + n_2u + n_3(v - v_c) + n_5e) + m_3\psi(\hat{v}) + m_5(\hat{f} - \hat{s})\right)(e_1 - e)\,. \tag{2.25}$$

The state variables are, respectively, the man–made capital–output ratio, employment ratio, unit wage, natural capital–output ratio, indicated natural capital–output ratio, gross unit rent, and unit depreciation of the natural capital. The requirement for the denominators to be positive is omitted. If $\dot{K} > 0$, $\dot{F} > 0$ at each instant of time, the system (2.19) – (2.25) defines *a strongly sustainable development.*

A non–trivial stationary state is defined as

$$\mathrm{E_a} = (s_a, v_a, u_a, f_a, c_a, y_a, e_a), \tag{2.26}$$

where

$$s_a = s_0,$$
$$v_a = (g + (1 - b - q)(d - n))/r,$$
$$u_a = (d - n - n_1 - n_3(v_a - v_c) - e_a n_5)/n_2,$$
$$f_a = (1 - u_a - e_a)/d - s_a/k,$$
$$c_a = f_a,$$
$$y_a = e_a + df_a,$$
$$e_a = e_1,$$
$$i = d.$$

At this stationary state, a growth rate of produced fixed capital, indicated natural capital, real natural capital, net output is the same: $\hat{K}_a = \hat{X}_a = \hat{F}_a = \hat{P}_a = d = \frac{m_1}{1-m_2-m_5} + n$. The stationary average profit rate is $(1 - u_a - y_a)/s_a = d/k$. The stationary rate of growth of real wage, labor productivity and capital intensities is $\hat{w}_a = \hat{a}_a = K_a \hat{/} L_a = F_a \hat{/} L_a = d - n$.

Kaldor's stylized facts on economic growth in industrialized capitalist economies are valid for this stationary state (Kaldor 1957). The requirements of the FENEG are also satisfied:

$K_a \dot{/} L_a = k(1 - u_a - y_a)a_a - nK_a/L_a.$

The higher the growth of labor force (n), the higher are the stationary rates of economic growth, stationary average profit rate and the faster is capital accumulation, like in the neo–classical model above. Thus, the importance of the rate of growth of labor force is the shared view in different streams of economic thought.

The form of the technical progress function (2.4) deserves a special attention. It has a special element, the function $\psi(\hat{v})$ that reflects the economy of scale. For $\hat{v} = 0$ and $[ABS\ (\hat{v})\wedge j]' = j[ABS\ (\hat{v})\wedge (j-1)]$, partial derivatives of the function $\psi(\hat{v})$ go to infinity, if $0 < j < 1$. The system (2.19) – (2.25) cannot be linearly approximated at the stationary state $E_a = (s_a, v_a, u_a, f_a, c_a, y_a, e_a)$ because partial derivatives of a Jacobian evaluated at this non–trivial stationary state go to infinity due to the same reason. As a rule, the stationary state E_a is not locally stable unlike the neo–classical stationary state. So the real economy cannot be observed in this state. Still it is possible to have periodic or quasi–periodic solutions of the system (2.19) – (2.25) that are bounded in the phase space.

For taking into account measurement errors and an impact of factors neglected in the model assumptions, the deterministic model (2.19) – (2.25) has been transformed in a stochastic model. Whereas the model (2.19) – (2.25) abstracts in particular from short-term and middle-term economic fluctuations, this stochastic model makes implicit allowances for them by specification of the random components. The latter model includes state equations and measurement equations

$$\mathbf{x}(n) = \mathbf{f}\,[\mathbf{x}(n-1)] + \mathbf{w}(n),$$
$$\mathbf{z}(n) = \mathbf{H}\mathbf{x}(n) + \mathbf{v}(n),$$

where $n = 1, 2,\dots N$ is an index of data samples, $\mathbf{x}(0)$ – a vector of an initial state of the system, $\mathbf{w}(n)$ – a vector of equations errors (driving noise), $\mathbf{v}(n)$ – a vector of measurement errors. The deterministic part $\mathbf{x}(n) = \mathbf{f}[\mathbf{x}(n-1)]$ corresponds to the system (2.19) – (2.25) and an additional integral equation for labor productivity a = INTEG ($\dot{a}$, a_0). The symbol $\mathbf{H}$ is for a rectangular matrix.

A simplified version of an extended Kalman filtering (EKF) applied assumes that all the multivariate moments of the second order equal zero. It assumes additionally that each of the random vectors $\mathbf{x}(0)$, $\mathbf{w}(n)$, $\mathbf{v}(n)$ has a constant mathematical expectation and dispersion. The covariance matrices (Ψ, $\mathbf{Q}$, $\mathbf{R}$) of these vectors are diagonal and invariable. Each element on the main diagonal is the dispersion of the respective stochastic component, all other matrix elements equal zero.

An application of the EKF to the U.S. macroeconomic data 1958–1991 has identified unobservable components of this stochastic model (Ryzhenkov 2001, 2002a). The reader can find the model parameters values in the Appendix.

It has been shown that long wave is a dominant non–equilibrium quasi–periodic behavioral pattern of the U.S. capital accumulation. Evaluating the historical fit through appropriate summary statistics and long–range forecasting has strengthened confidence in this model. In an exploratory scenario, a spiral of accumulation is almost periodically arrested by the relative shortage of labor. A quasi–period of fluctuations is about 29–33 years.[7] This duration is shorter than earlier estimations of the period of long wave (Forrester 1992; Sterman 1985, 1986, 1990). The reduction of the long wave's period

[7] Roughly the same estimations for the period of the economic long wave in the USA are given in the books (Chizhov 1977: 110-124), (Gerster 1988) and paper (Kiefer 1996).

may be explained by shortened product life cycles, resource intensive R&D and some other factors, analyzed in (Milling 2002).[8]

The current downswing in the long wave manifests itself in the growing produced capital–output ratio and unit wage, declining profitability and employment ratio. There is a secular profit squeeze and deceleration of economic growth in spite of the steady reduction of the eco–intensity and labor productivity growth. Worsening profits slow the growth in productivity that inhibits profits, in turn. The both profit rates $(1 - u - e)/(s + f)$ and $(1 - u - y)/s$ tend to be lower and lower than the benchmark $d/k \approx 0.144$ in an exploratory scenario.

A shorter period of simulations (until the year 2034) provides us with a more detailed picture of the long wave in the first third of the XXI century.

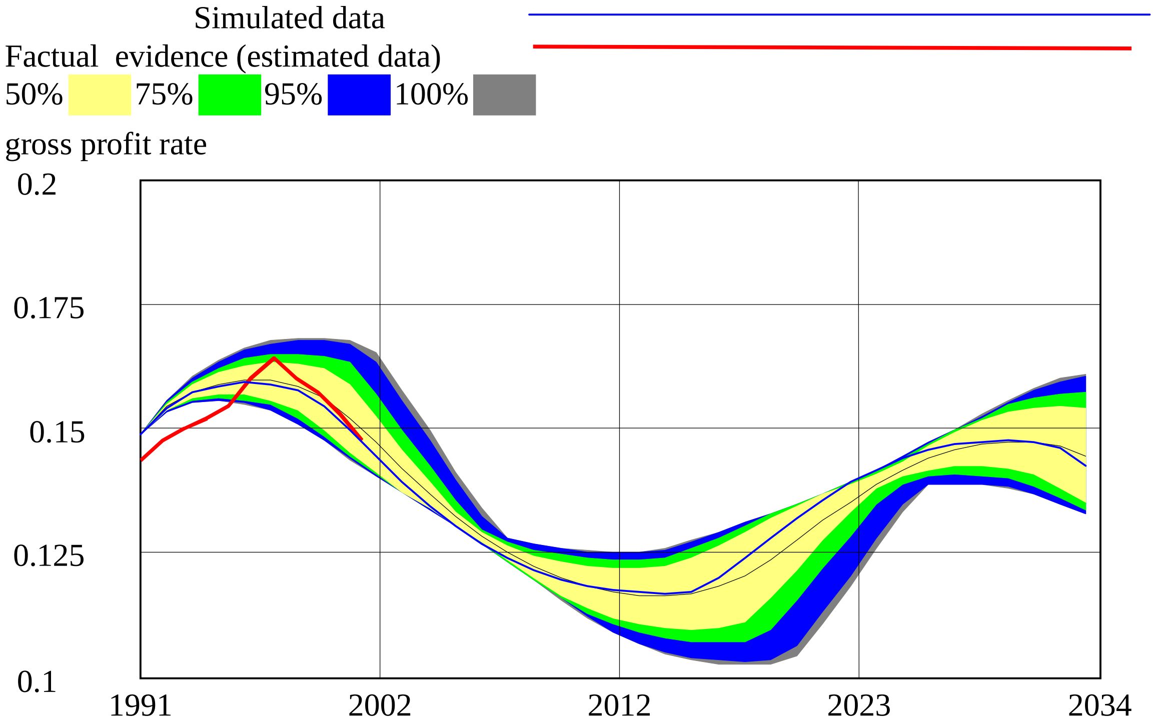

Figure 2 Confidence bounds for the gross profit rate, $(1 - u)/s$, in the USA, 1991–2034 (the exploratory scenario) compared with the factual evidence for 1991-2001

The initial state vector of the above stochastic model for the year 1991 has been estimated by the EKF based on the statistical information over 1958-1991 given in (Ryzhenkov 2001).

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[8] A review *Innovation in Industry* works out that the economic long waves are shortening from 50–60 years to around 30–40 years. See: *The Economist*, February 20th 1999, 350 (8107): 8. Numbered 1–5, these long waves correspond to the industrial revolutions.

It is assumed for simplicity that the control parameter (*j*) in the modified technical progress function (2.4) is randomly uniformly distributed in the interval (0.111, 0.311) with the variance about 0.0033. Two hundreds of Monte Carlo simulations with the initial noise seed of 1234 have been calculated and compared with the factual evidence on Figures 2 and 3. The factual evidence, given here by the author, is based on the official U.S. statistics.

The simulations display the confidence bounds for 1991-2034. These bounds are computed at each point in time by ordering and sampling all the simulation runs. For example, for a confidence bound at 50, a quarter of the runs have a value lower than the top of the confidence bound and another quarter of the runs have a value higher than the bottom. The graph's tread displays the change of the mean value over time.

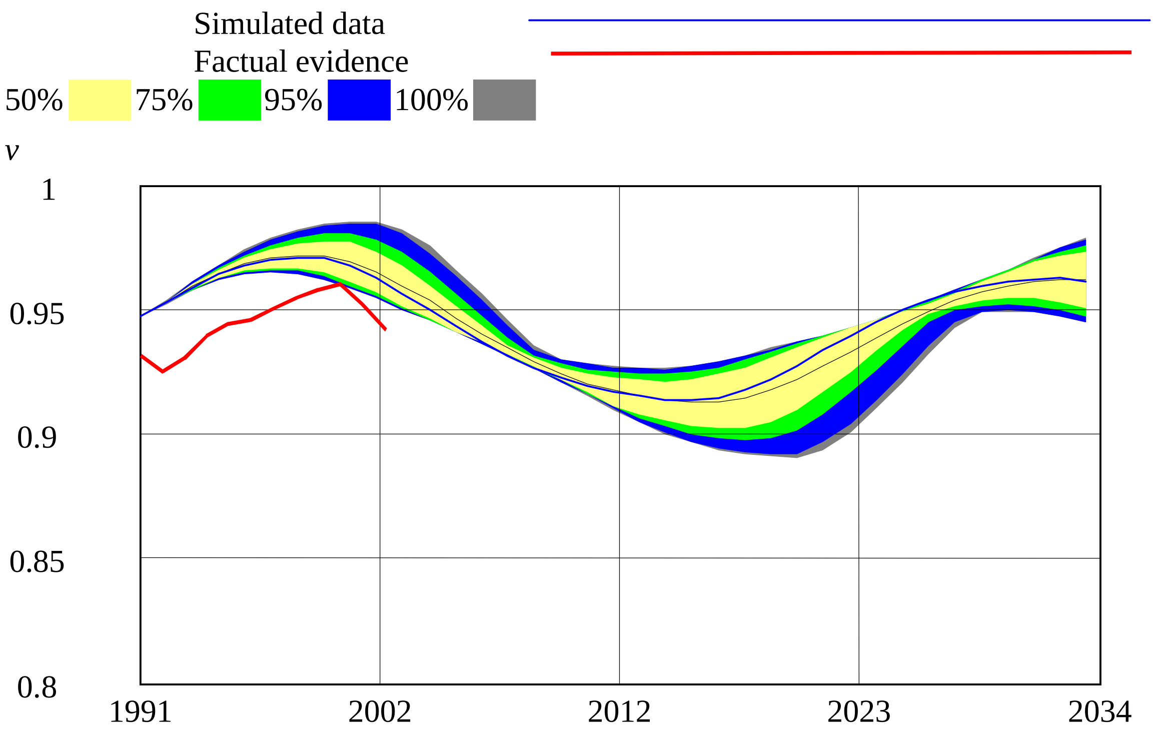

Figure 3 Confidence bounds for the employment ratio (v) in the USA, 1991–2034 (the exploratory scenario) compared with the factual evidence for 1991-2002

The long-term business upturn will not probably happen until 2012 or even 2018. It will proceed thereafter up to the beginning of the next long-term downturn in 2035-2040.

The real development differs from the offered description because of learning, external influences and counter-cyclical policies that are not taken into account. Still the model parameters can be adjusted by EKF and the forecast can be updated each period, based on new information.

The conscious element of the HL may play a decisive role in providing better governance of the ecological–economic reproduction on the increasing scale when ecology remains one of the major political issues.

2.3 The normative scenario 1991–2107: extending the natural capital

The second scenario corresponds to a rather strong criterion of sustainable development ($\dot{K} > 0$, $\dot{F} > 0$ for $t > 2003$). In particular, the society deliberately raises gross unit rent step–wise in the year 2003:

$$\dot{y} = (o_1(c - f) + o_2 \hat{f})y + \text{STEP}(0.0018, 2003). \qquad (2.24a)$$

At the end of the year 2002 or beginning of 2003, $y_{2003} \approx 0.00656$; at the end of the year 2003 or beginning of 2004, $y_{2004} \approx 0.00757$. This modification does not exclude other possible alterations for achieving sustainable development. Still it addresses the critical shortcoming of the exploratory scenario, namely the depletion of the natural capital.

In the normative scenario, the economic growth is quasi–cyclical with a period of about 31–33 years. The maximum employment is firstly achieved in the year 1999, it declines thereafter until the year 2011, then it grows again until the year 2028. The increase in the gross unit rent is achieved by a reduction of the unit wage by about the same quantity. Still this partial redistribution of the NNP produces desirable positive effects over the whole period on the average:

- the rate of the economic growth rate is increased;
- the natural capital is extended;
- the average and general rates of profit are raised without any apparent tendency to fall;
- there are gains in the employment rate;
- the labor productivity and real wage of an employee increase faster than in the previous (exploratory) scenario.

So far investing in natural capital has had a lower profitability than investing in produced capital in the modern capitalist economy. The society can overcome this market failure by an appropriate policy as suggested. Still some additional considerations are required.

2.4 Expected dynamics of American labor force up to 2050

In the exploratory and normative scenarios, the rate of growth of the labor force is constant. It has been estimated based on the information for the basal period, 1958–1991. Figure 4 shows substantial deviations between the observed and ‘naively’ extrapolated back and forth growth rates of the labor force. A more substantial divergence is expected in the future. The U.S. Administration projects the American labor force to grow at a 1.0 percentage average pace over 2001 to 2012.[9] This rate of growth does not take into account changes in hours worked annually per worker.

[9] Economic Report of the President. 2002. United States Government Printing Office: Washington (DC): 55.

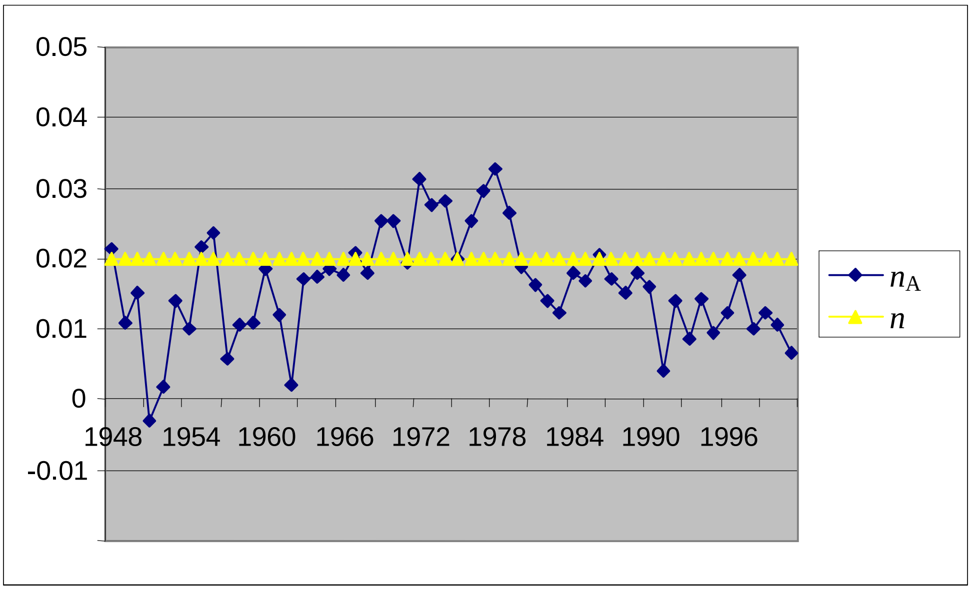

Figure 4 The observed (n_A) and estimated (n) growth rates of labor force in the USA, 1948–2001[10]

This slowdown in the growth rate of the labor force affects the proposed policy for achieving sustainable development in the normative scenario (2.24a). Computer simulations based on the HL of capital accumulation (2.19) – (2.25) have shown that the increase in the gross unit rent (y) by 0.18 percentage point as in (2.24a) is not sufficient if the growth rate of the labor force (n) equals 1 per cent a year over the whole period until the year 2050. The economic-ecological reproduction becomes non-sustainable and its scale decreases.

The reason is that due to the slower growth of the labor force and labor productivity, economic growth and accumulation of capital decelerate (Figure 5). *Ceteris paribus*, the eco–intensity (e) is the higher, the lower is the growth rate of the labor force (n). With the general economic slowdown the absolute rate of decline of the unit ecological damage (e) becomes smaller, therefore greater environmental investment is required for strongly sustainable development.

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[10] The actual growth rates of labor force are calculated based on Economic Report of the President. 2002. United States Government Printing Office: Washington (DC): Table B–35. The estimated growth rate (n) for 1958–1991 is from (Ryzhenkov 2001, 2002a).

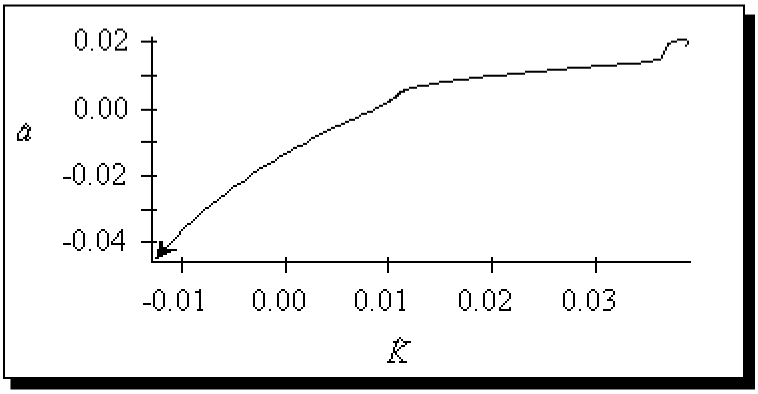

Figure 5 The declining growth rates of produced fixed capital ($\hat{K}$) and labor productivity ($\hat{a}$) in 1991-2050 for n = 0.01

Projecting the growth rate of labor force as equal 1% per year for the whole forecasting period until 2107 is, probably, unrealistically high. The work done by the U.S. Bureau of Labor Statistics (BLS) helps to be more precise.

The BLS projections depend on assumptions of the future size and composition of the current population, as well as on the trends in labor force participation rates of different population groups. The Table 1 reflects the long-term slowdown in growth of population and labor force.

Table 1 Annual growth rates of the civilian non-institutional population, civilian labor force, and civilian labor force participation rate, 1990-2000, and projected, 2000 to 2500 (% a year)

Category	1990-2000	2000-10	2010-15	2015-20	2020-30	2030-40	2040-50
Population growth	1.0	1.1	0.8	0.8	0.8	0.7	0.6
Participation growth	0.12	0.05	-0.2	-0.53	-0.43	-0.11	-0.02
Interaction[1]	-0.02	-0.05	0	-0.07	-0.07	0.01	0.02
Labor force growth (constant hours worked a year per head)	1.1	1.1	0.6	0.2	0.3	0.6	0.6

[1] Interaction measures effects of changing composition of labor force, in particular, due to aging and death. Interaction is the labor force growth that is not accounted for by growth in the aggregate population and aggregate labor participation rate. Source: Toosi 2002: 18.

In the 1990s, the growth rate of the labor force exceeded that of the population. This positive gap narrows and will close entirely by 2010. In the latter period (2010-2040) this gap will become negative. It will disappear in the latest decade (2040-2050) again.

3 Upgrading the model of sustainable development

Ascending from abstract to concrete requires a further theoretical elaboration of the HL paying attention to factors behind the growth of labor force. There are at least two technical ways for presenting a changeable growth rate of labor force. The first way is adding an auxiliary equation for this rate to the model. The second way is extending the initial model by a new differential equation for the new state variable – the growth rate of labor force. The latter is likely more powerful: it allows increasing the dimensionality of the phase state for reflecting complex forms of a socio-economic evolution in an extended model. Only the research can find a mostly appropriate partial dynamic law for the growth rate of labor force and grasp it in a more general law of capital accumulation than proposed so far.

On the present stage of research, the growth rate of labor force is an exogenous auxiliary variable modeled with a help of the *Powersim* built-in STEP function:

$$n = \text{IF}(\text{TIME}<2010,\ 0.011,\ 0.006) + \text{STEP}(-0.004,\ 2015) + \text{STEP}(0.001,2020) + \text{STEP}(0.003,\ 2030).$$

This presentation uses data from Table 1. It is assumed that the growth rate of the labor force (n = 0.006) will not change in the period 2030-2107.

The decelerating uneven growth of the American labor force challenges U.S. strongly sustainable development that requires permanent accumulation of man-made capital and natural capital. Extending natural capital is especially problematical.

After many simulations experiments with different investment policies aimed at strongly sustainable development, a new equation for the rate of change of the gross unit rent has been found that enables a positive net unit rent (y – e) in 2005-2107 (Figure 6):

$$\dot{y} = (o_1(c - f) + o_2\hat{f})y + \text{STEP}(0.0032,2003) + \text{STEP}(0.002,2010) + \text{STEP}(0.002,2015) + \text{STEP}(0.002,\ 2030) + \text{STEP}(0.002,2075) + \text{STEP}(0.002,2100).$$

This investment policy is favorable for securing strongly sustainable development, since both the man-made capital (K) and natural capital (F) grow permanently in this period (Figure 7). The employment ratio tends to increase; the secular trend of the average profit rate to fall is moderated (Figure 8).

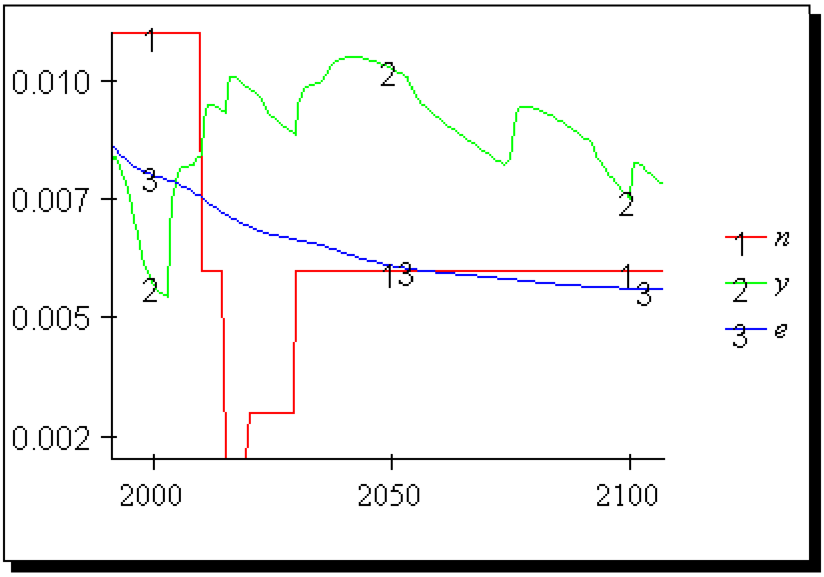

Figure 6 The gross unit rent (y), unit depletion and degradation of natural capital (e) and growth rate of labor force (n), 1991-2107

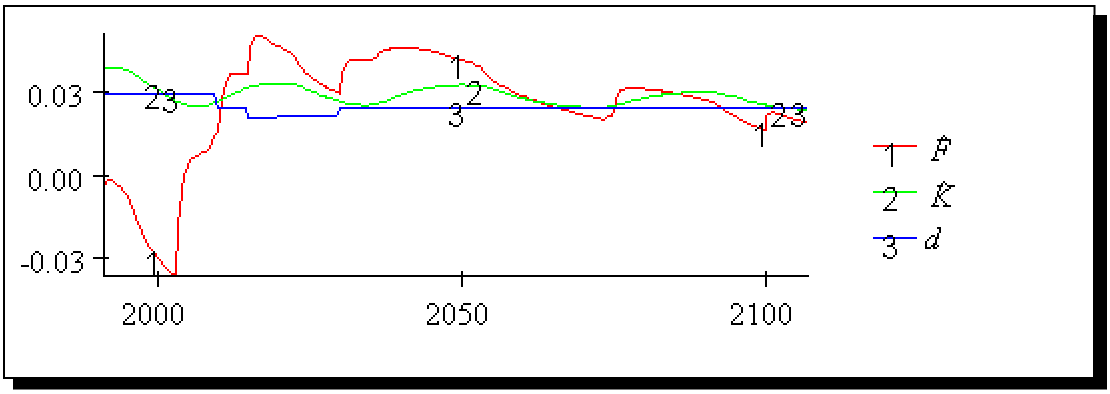

Figure 7 A transition to strongly sustainable development after the year 2004: the growth rates of natural capital ($\hat{F}$) and man-made capital ($\hat{K}$) versus the benchmark (d) in 1991-2107

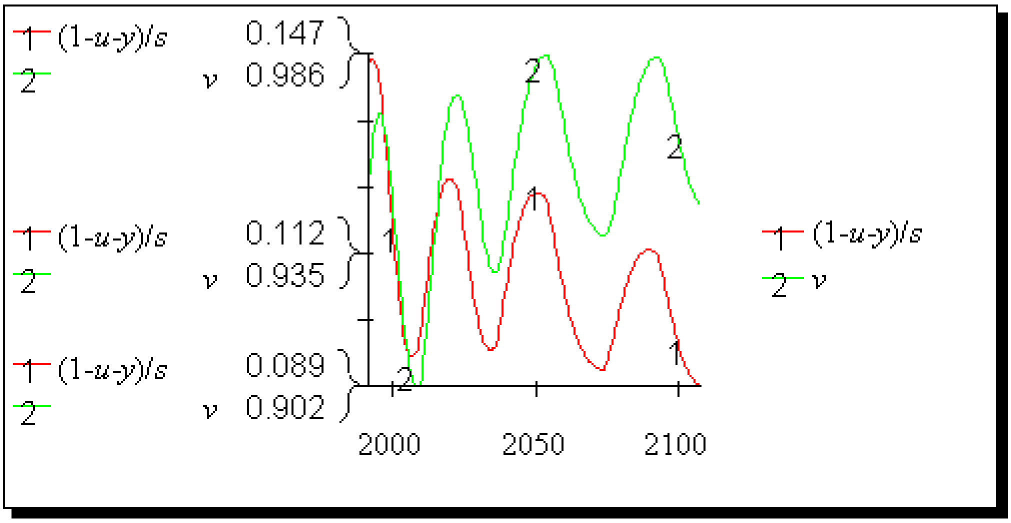

Figure 8 The average profit rate, (1-*u*-*y*)/*s*, and employment ratio (*v*) in 1991-2107

The accumulation of natural capital facilitates the growth of population and labor force that, in turn, reinforces the economic-ecological reproduction. An explicit modeling of this positive feed-back would require a treatment of the growth rate of labor force as endogenous variable.

A. Okun (Okun 1983: 154) wrote: "…the postwar record has convincingly delivered the verdict that a weak labor market depresses the size of the labor force. But the magnitude and timing of the effect is not clear … The response of participation rate is likely to be a complicated lagged phenomena which will not be closely tied to the current unemployment rate." The author will advance research of these complex phenomena in a future research.

This research would be facilitated by improvement of the official statistics of labor force. So far it has not been tracking the components of change of labor force similar to the components of change of the American population. This incompleteness does not make easier a grasping of the growth rate of labor force as an endogenous variable.[11]

Conclusion

This paper has elaborated the notion of viable quasi-periodic motion bounded in the phase space that generalizes the notions of stationary growth and stationary cyclical growth, which are presented by a point and limit cycle in the phase space, respectively. This elaboration has been supported by the simulation experiments, based on the HL of capital accumulation, and by statistical data. The long wave has been exposed not as long-term fluctuations around an equilibrium trend but as a quasi-periodic non-equilibrium trend and stochastic attractor. This theoretical presentation differs fundamentally from the neo-classical view on economic growth as a convergence

[11] The official U.S. statistics of population does track the components of change due to death, birth and net immigration.

towards equilibrium. The fundamental equation of neo-classical growth is a special case of the more general dynamic regularity, presented above as the direct consequence of the HL. Refining the given formulation of this law requires treatment of the growth rate of labor force as endogenous parameter. This puzzle, still unsolved, is left for future research.

The deceleration of labor force growth challenges U.S. sustainable development in the XXI century. The application of the HL with exogenous growth of labor force to the American economy has shown that the moderation of the secular tendency of the average profit rate to fall is conditioned by the society's investment strategies. This moderation could be even more successful if enhanced investments in natural capital are combined with appropriate political innovations for transiting to natural capitalism.

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Appendix

In model simulations and forecasts, the Runge-Kutta integration of the fourth order with a automatically chosen time step have been used. This technique provides approximation to the underlying continuous system. According to the Powersim and Vensim manuals, RK4 Auto is the most accurate technique to use. It performs, automatically, the experiment of decreasing TIME STEP to assure accuracy. If accuracy is below an acceptable tolerance, the integration interval is decreased further until the desired accuracy is obtained.

The initial vector used in forecasts is an EKF sub-optimal estimator of the phase vector of the stochastic model for the year 1991 the based on statistical information for 1958-1991: $a_0 \approx 0.0512$, $c_0 \approx 13.98$, $e_0 \approx 0.0087 > e_a = e_1 \approx 0.005$, $f_0 \approx 0.1091$, $s_0 \approx 2.052$, $u_0 \approx 0.695 > u_a \approx 0.645$, $v_c \approx 0.925 < v_0 \approx 0.948 < v_a \approx 0.991$, $y_0 \approx 0.008$.

The EKF has allowed to get also suboptimal magnitudes of the model parameters: the coefficients of the deterministic part of the stochastic model, diagonal elements of the matrices $\mathbf{Q}$, Ψ and $\mathbf{R}$. The indexes of these diagonal elements coincide with letters used for presenting the variables themselves.

The VENSIM Optimisation Files

The following specifications of the noise and optimization pay-off are in agreement with the VENSIM format:

pay-off.vpd

a/ $\mathbf{R}_a$
e/ $\mathbf{R}_e$
f/ $\mathbf{R}_f$
s/ $\mathbf{R}_s$
u/ $\mathbf{R}_u$
v/ $\mathbf{R}_v$
y/ $\mathbf{R}_y$

kalman.prm

a/ $\mathbf{Q}_a$ / Ψ_a
c/ $\mathbf{Q}_c$ / Ψ_c
e/ $\mathbf{Q}_e$ / Ψ_e
f/ $\mathbf{Q}_f$ / Ψ_f
s/ $\mathbf{Q}_s$ / Ψ_s
u/ $\mathbf{Q}_u$ / Ψ_u
v/ $\mathbf{Q}_v$ / Ψ_v
y/ $\mathbf{Q}_y$ / Ψ_y

For the unobservable variable c $\mathbf{Q}_c = 0$ and $\Psi_c = 1$ by the Vensim default. All other variances from the file *kalman.prm* and variances of the measurement noise (*pay-off.vpd*) for the seven observable state variables have been included in the list of parameters to be estimated. The *File 2.out* contains the best payoff so far, the reason the optimiser stopped, and the values of the search parameters needed to achieve that payoff.

File 2.out

:COMSYS After 2123 simulations
:COMSYS Best payoff is 930.511
:COMSYS User terminated multiple search session
:OPTIMISER = Powell
:SENSITIVITY = Payoff Value
:MULTIPLE_START = Random
:RANDOM_NUMER = Linear
:OUTPUT_LEVEL = 2
:TRACE = 2

:MAX_ITERATIONS = 10000
:PASS_LIMIT = 2
:FRACTIONAL_TOLERANCE = 0.0003
:TOLERANCE_MULTIPLIER = 21
:ABSOLUTE_TOLERANCE = 1
:SCALE_ABSOLUTE = 1
:VECTOR_POINTS = 1.24418e–306

$0.125 \leq$ TIME STEP $= 0.588893 \leq 3$

$0 \leq b = 0.621019 \leq 1$

$0 \leq c_0 = 13.2528$

$0 \leq e_1 = 0.0054249$

$0 \leq g = 0.0531989 \leq 1.5$

$i = 0.0373606$

$0.05 \leq j = 0.211049 \leq 1$

$0.2 \leq k = 0.267234 \leq 0.5$

$0 \leq m_1 = 0.0149761 \leq 0.02$

$0.1 \leq m_2 = 0.1 \leq 0.75$

$0 \leq m_3 = 0.0105916 \leq 0.1$

$0 \leq m_5 = 0.0888489 \leq 0.3$

$0 \leq n = 0.0199143 \leq 0.022$

$n_1 = -0.24223 \leq 0.02$

$0 \leq n_2 = 0.353022 \leq 0.5$

$0 \leq n_3 = 0.5 \leq 0.5$

$0 \leq n_5 = 0.0106352 \leq 1$

$o_1 = -0.0299728$

$o_2 = -9.93389$

$q = -0.0084833$

$0 \leq r = 0.0609304$

$0.75 \leq v_c = 0.92536 \leq 0.99$

$1e{-}009 \leq \mathbf{Q}_a = 2.93202e{-}007 \leq 0.001$

$\mathbf{Q}_c = 1$

$1e{-}009 \leq \mathbf{Q}_e = 1.22235e{-}007 \leq 1e{-}006$

$1e{-}009 \leq \mathbf{Q}_f = 1e{-}009 \leq 0.001$

$1e{-}009 \leq \mathbf{Q}_s = 0.000929877 \leq 0.01$

$1e{-}009 \leq \mathbf{Q}_u = 5.51477e{-}009 \leq 0.001$

$1e{-}009 \leq \mathbf{Q}_v = 5.16068e{-}007 \leq 0.001$

$1e{-}009 \leq \mathbf{Q}_y = 3.9415e{-}006 \leq 2e{-}005$

$1.25e{-}006 \leq \mathbf{\Psi}_a = 1.25e{-}006 \leq 5e{-}006$

$\mathbf{\Psi}_c = 0$

$5e{-}006 \leq \mathbf{\Psi}_e = 5e{-}006 \leq 2e{-}005$

$0.001 \leq \mathbf{\Psi}_f = 0.001 \leq 0.004$

$0.0234 \leq \mathbf{\Psi}_s = 0.0234 \leq 0.0936$

$0.0025 \leq \mathbf{\Psi}_u = 0.0025 \leq 0.01$

$0.00435 \le \Psi_v = 0.00435 \le 0.0174$
$5e{-}006 \le \Psi_y = 5e{-}006 \le 2e{-}005$
$1.25e{-}006 \le \mathbf{R}_a = 1.25e{-}006 \le 5e{-}006$
$5e{-}006 \le \mathbf{R}_e = 5e{-}006 \le 0.0002$
$0.001 \le \mathbf{R}_f = 0.001 \le 0.004$
$0.0234 \le \mathbf{R}_s = 0.0234 \le 0.0936$
$0.0025 \le \mathbf{R}_u = 0.0025 \le 0.01$
$0.0043 \le \mathbf{R}_v = 0.0043 \le 0.0174$
$5e{-}006 \le \mathbf{R}_y = 3.41319e{-}005 \le 0.0002$

The stationary state E_a with almost full employment in a literal sense ($v_a \approx 0{,}991$) is not stable, unlike a full employment stationary state in the Solow model.

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