Eigenvalue and Eigenvector Analysis of Dynamic Systems

Paulo Goncalves*

Abstract

While several methods aimed at understanding the causes of model behavior have been proposed in recent years, formal model analysis remains an important and challenging area in system dynamics. This paper describes a mathematical method to incorporate eigenvectors to the more traditional eigenvalue analysis of dynamic models. The proposed method derives basic formulas that characterize how a change in link (or loop) gain influence state behavior in linear dynamic systems. Based on the insights developed from linear theory, I extend the method to nonlinear dynamic systems by linearizing the system at every point in time and evaluating the impact to the derived formulas. The paper concludes with an application of the method to a linear system.

1. Introduction

Formal model analysis remains an important and challenging area in system dynamics. Several methods aimed at understanding the causes of model behavior have been proposed in recent years (Kampmann 1996; Mojtahedzadeh 1997; Gonçalves, Lertpattarapong and Hines 2000; Saleh and Davidsen 2001; Saleh 2002; Mojtahedzadeh, Richardson and Andersen 2004; Oliva 2004; Oliva and Mojtahedzadeh 2004; Güneralp 2005; Hines 2005; Kampmann and Oliva 2005; Saleh, Davidsen and Bayoumi 2005). These methods trace back two threads in model analysis: the loop dominance work of Richardson (1995) and eigenvalue elasticity work of Forrester (1982). Mojtahedzadeh (1997) and Mojtahedzadeh, Richardson and Andersen (2004) extend the loop dominance work first proposed by Richardson (1995). The research proposes pathway participation metrics (PPM) to find the structure that most influences the time path of a given variable. The method provides a local assessment of how changes in a state variable of interest influence the net change of the same variable ($d\dot{x}_k / dx_k$). While the method has the advantage of being computationally simple it is not well suited for systems that oscillate, since the analysis is local and cannot capture global modes of behavior.

Most of the remaining research traces back to eigenvalue elasticity theory proposed by Forrester (1982). The method calls for the computation of eigenvalues and then explores how the eigenvalues change as link gains change, that is, link gain elasticities. Forrester showed that a complete description of *link* elasticities allows one in principle to calculate *loop* elasticities. This suggestion though never implemented in software, promised to provide an answer to how model structure, that is a set of feedback loops, determines model behavior. The particular calculation that Forrester suggested is actually not feasible. As he realized later, Forrester's suggested approach results in a system of equations that is over-determined – an effect of the fact that the number of loops increases much faster than the number links. Kampmann discovered that a small subset of loops is sufficient to uniquely describe eigenvalues (i.e. the behavior) of a system dynamics model (Kampmann 1996). Using an Independent Loop Set (ILS) produces a smaller system of equations, a system that can be solved. The Independent loop set (ILS) method has the important advantage of allowing us to calculate loop gains from link gains, where the number of links in a model is often small. However, it has the disadvantage of relying on an *ad hoc* procedure to select the independent loop set (ILS). Gonçalves, Lertpattarapong and Hines (2000) use Mason's rule to express the characteristic equation and its solutions (eigenvalues) in terms of loop gains (instead of link gains), which allows them to obtain *loop* gain elasticities directly. While the method sidesteps the problems associated with an arbitrary selection of loops it has the shortcoming of requiring the computation of all loops in the model, a number that rises quickly even with moderate size models. Oliva (2004) provides an extension to the method selecting first the shortest loops. The shortest independent loop set (SILS) provides a systematic representation of the feedback complexity in its simplest components and it is the most granular description of the structure in a cycle partition. Oliva and Mojtahedzadeh (2004) compare the results obtained with the SILS approach to that of PPM and find that the loops generating the main dynamics are often included in the SILS. More recently, Kampmann and Oliva 2006 explore the application of loop eigenvalue elasticity to three models to assess the potential of the method and find that the insights depend on the character and dynamics of the model.

The work of Saleh, Davidsen and Bayoumi (2005) is most akin to ours in its interest in understanding the contribution of both eigenvalues and eigenvectors on model behavior. While we focus on the analytical computation of the influence of eigenvalues and eigenvectors on model behavior, Saleh et al. (2005) provide a computational method (implemented in Matlab) to calculate such influence. The motivation for this paper is to provide a mathematical framework for future work on eigenvector and eigenvalue analysis. This work follows the research tradition

* Assistant Professor, Management Science Department, School of Business Administration, University of Miami, Coral Gables, FL 33124, USA. Phone: (305) 284-8613/Fax: (305) 284-2321. paulog@miami.edu

of Forrester (1982). Similarly to previous research, our interest between structure and behavior is expressed in terms of understanding how changes in links or loops gain affect the time path behavior of a state variable. Our work departs from previous efforts in terms of its focus on analytical results and emphasis on the impact that first time derivatives of eigenvalues and eigenvectors have on model behavior, instead of eigenvalue elasticities.

2. Behavior in Linear Dynamic Systems

The formal structure of a linear system dynamics model with a vector of state variables $\mathbf{x}$(t), where $\mathbf{x}$(t) = ($x_1, x_2, \ldots, x_n$)', a vector of first time derivatives of the state variables $\dot{\mathbf{x}}$(t), where $\dot{\mathbf{x}}$(t) = ($\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n$)', a gain matrix $\mathbf{J}$ capturing the partial derivatives of the net change of a state variable with respect to another (the matrix $\mathbf{J_{nxn}} = \partial\,\dot{\mathbf{x}} / \partial\,\mathbf{x}$ is commonly known as the *Jacobian* of the system), and a constant vector **b,** can be represented compactly in the following way:

$$\dot{\mathbf{x}} = \mathbf{J}\mathbf{x} + \mathbf{b} \tag{1}$$

Consider now the solution to the homogeneous system. A standard result in linear systems theory is that the eigenvalues (λ) of the matrix $\mathbf{J}$ describe the behavior modes inherent in the model and are the solutions of the characteristic polynomial ($P(\lambda)$), where ($P(\lambda) = \left|\lambda I_n - \mathbf{J}\right| = 0$). Assume for simplicity that the system matrix $\boldsymbol{J_{nxn}}$ has a complete set of n linearly independent eigenvectors ($\boldsymbol{r_1, r_2, \ldots, r_n}$) with corresponding eigenvalues ($\lambda_1, \lambda_2, \ldots, \lambda_n$), where eigenvalues may or may not be distinct. Since the eigenvectors are linearly independent, they span the n dimensional space, therefore an arbitrary value of the state $\boldsymbol{x}(t)$ can be expressed by the linear combination of the eigenvectors:

$$\mathbf{x}(t) = z_1(t)\mathbf{r_1} + z_2(t)\mathbf{r_2} + \ldots + z_n(t)\mathbf{r_n} \tag{2}$$

where $z_i(t)$, $i=1, 2, \ldots, n$ are scalars.

Using the fact that by definition multiplication of the system matrix by their eigenvectors results in the product of the eigenvectors by eigenvalues ($\boldsymbol{J r_i} = \lambda_i \boldsymbol{r_i}$), we can rewrite equation (2) by multiplying it by the system matrix $\boldsymbol{J_{nxn}}$.

$$\mathbf{J}\mathbf{x}(t) = \dot{\mathbf{x}}(t) = z_1(t)\mathbf{J}\mathbf{r_1} + z_2(t)\mathbf{J}\mathbf{r_2} + \ldots + z_n(t)\mathbf{J}\mathbf{r_n}$$

$$\dot{\mathbf{x}}(t) = z_1(t)\lambda_1\mathbf{r_1} + z_2(t)\lambda_2\mathbf{r_2} + \ldots + z_n(t)\lambda_n\mathbf{r_n} \tag{3}$$

Since equation (2) defines the state vector $\boldsymbol{x}(t)$, we can take its first time derivative. In addition, using the fact that eigenvalues and eigenvectors are constant in linear systems, we can rewrite (2) to get:

$$\dot{\mathbf{x}}(t) = \dot{z}_1(t)\mathbf{r_1} + \dot{z}_2(t)\mathbf{r_2} + \ldots + \dot{z}_n(t)\mathbf{r_n} \tag{4}$$

Comparing the right hand side of (4) and (3), we obtain:

$$\dot{z}_1(t)\mathbf{r_1} + \dot{z}_2(t)\mathbf{r_2} + \ldots + \dot{z}_n(t)\mathbf{r_n} = z_1(t)l_1\mathbf{r_1} + z_2(t)l_2\mathbf{r_2} + \ldots + z_n(t)l_n\mathbf{r_n} \tag{5}$$

And since the eigenvectors are linearly independent, the equality can only hold if:

$$\dot{z}_i(t) = z_i(t)l_i \tag{6}$$

The system above can be represented in matrix form as: [1]

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \\ \ldots \\ \dot{z}_n(t) \end{bmatrix} = \begin{bmatrix} l_1 & 0 & \ldots & 0 \\ 0 & l_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & l_n \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \\ \ldots \\ z_n(t) \end{bmatrix} \tag{7}$$

The solution of the homogeneous system of decoupled equations presented above is known:

$$\begin{bmatrix} z_1(t) \\ z_2(t) \\ \ldots \\ z_n(t) \end{bmatrix} = \begin{bmatrix} e^{l_1} & 0 & \ldots & 0 \\ 0 & e^{l_2} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & e^{l_n} \end{bmatrix} \begin{bmatrix} z_1(0) \\ z_2(0) \\ \ldots \\ z_n(0) \end{bmatrix} \text{ or } z_i(t) = e^{l_i t} z_i(0) \tag{8}$$

Substituting the result in (8) in our original equation (2) yields: [2]

$$\mathbf{x}(t) = e^{l_1 t} z_1(0)\mathbf{r_1} + e^{l_2 t} z_2(0)\mathbf{r_2} + \ldots + e^{l_n t} z_n(0)\mathbf{r_n} \tag{9}$$

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[1] Note that we rewrite the results above more compactly in matrix form defining $\boldsymbol{V}$ as the nxn matrix whose n columns are the eigenvectors of $\boldsymbol{J}$ and defining the column vector $\boldsymbol{z}$(t) with components ($z_1(t), z_2(t), \ldots$ $z_n(t)$). Defining $\boldsymbol{V}$ that way allows us to write equation (2) as $\mathbf{x}(t) = \mathbf{V}\mathbf{z}(t)$. We can interpret the new equation as a change in variable and use it to rewrite the dynamic system, which yields: $\mathbf{V}\dot{\mathbf{z}}(t) = \mathbf{J}\mathbf{V}\mathbf{z}(t)$ or simply: $\dot{\mathbf{z}}(t) = \mathbf{V}^{-1}\mathbf{J}\mathbf{V}\mathbf{z}(t)$, where the computation of the inverse of the matrix of eigenvectors ($\boldsymbol{V}^{-1}$) depends on the value of all the system eigenvectors. The new system ($\dot{\mathbf{z}}(t)$) is related to the original one ($\dot{\mathbf{x}}(t)$) by a change of variable. The new system matrix ($\boldsymbol{V}^{-1}\boldsymbol{J}\boldsymbol{V}$) corresponds to the system governing the $\boldsymbol{z}(t)$ state equations, where the change in each state ($\dot{z}_i(t)$) depends only on the product of the associated eigenvalue (l_i) and the own state ($z_i(t)$). Accordingly, we can write $\boldsymbol{V}^{-1}\boldsymbol{J}\boldsymbol{V}=L$, where L is the diagonal matrix with the eigenvalues of $\boldsymbol{J}$ in the diagonal.

[2] The initial values of $z(0)$ can be obtained in terms of $x(0)$ from the change in variable definition: $\mathbf{z}(0) = \mathbf{V}^{-1}\mathbf{x}(0)$.

3. How Links Influence System Behavior

We focus our attention on equation (9) to understand how changes in link gains (i.e., the strength of model parameters) influence system behavior. The behavior of each state in the system $x_i(t)$ can be described by:

$$x_i(t) = r_{1i} e^{l_1 t} z_1(0) + r_{2i} e^{l_2 t} z_2(0) + \ldots + r_{ni} e^{l_n t} z_n(0) \qquad (10)$$

where r_{1i} is the i-th component of the first eigenvector.

The equation suggests that the dominant behavior mode of the state variable $x_i(t)$ will be determined by the relative size of each i-th component of each eigenvector $\mathbf{r_j}$, where $j=1$ to n. We can rewrite equation (10) above in matrix form:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ \ldots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} r_{11} & r_{21} & \ldots & r_{n1} \\ r_{12} & r_{22} & \ldots & r_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ r_{1n} & r_{2n} & \ldots & r_{nn} \end{bmatrix} \begin{bmatrix} e^{?_1 t} z_1(0) \\ e^{?_2 t} z_2(0) \\ \ldots \\ e^{?_n t} z_n(0) \end{bmatrix} \qquad (11)$$

Equation (9) highlights that the behavior of each state is influenced both by eigenvalues (l_i) and eigenvectors (r_{ji}). In addition, both eigenvalues (l_i) and eigenvectors (r_{ji}) depend on the values of link gains (i.e., parameters in the model), because eigenvalues are solutions to the characteristic polynomial ($P(l)$), where $P(l) = \left| l I_n - \mathbf{J} \right| = 0$ and the entries of the *Jacobian* (***J***) are the partial derivatives or the link gains (a_{kl}) in a system dynamics model. Therefore, a change in the gain of an arbitrary link (a_{kl}) results in a new *Jacobian* and different values for both eigenvalues (l_i) and eigenvectors (r_{ji}). To understand the nature of the impact of changes in link gains on system behavior, we take the partial derivative of each state in the system $x_i(t)$ with respect to its link gains. From equation (10), we obtain the change in behavior of each state $x_i(t)$ due to changes in link gain (a_{kl}) as:

$$\frac{\partial x_i(t)}{\partial a_{kl}} = \frac{\partial}{\partial a_{kl}} \left[r_{1i} e^{?_1 t} z_1(0) + \ldots + r_{ni} e^{?_n t} z_n(0) \right] \qquad (12)$$

and taking the derivative of individual components, we obtain:[3]

$$\frac{\partial x_i(t)}{\partial a_{kl}} = \frac{\partial r_{1i}}{\partial a_{kl}} e^{?_1 t} z_1(0) + r_{1i}\frac{\partial e^{?_1 t}}{\partial ?_1}\frac{\partial ?_1}{\partial a_{kl}} z_1(0) + \dots + \frac{\partial r_{ni}}{\partial a_{kl}} e^{?_n t} z_n(0) + r_{ni}\frac{\partial e^{?_n t}}{\partial ?_n}\frac{\partial ?_n}{\partial a_{kl}} z_n(0) \quad (13)$$

We can rewrite equation (13) in a more compact way as:

$$\frac{\partial x_i(t)}{\partial a_{kl}} = \sum_{j=1}^{n}\left(\frac{\partial r_{ji}}{\partial a_{kl}} e^{?_j t} + r_{ji}\frac{\partial e^{?_j t}}{\partial ?_j}\frac{\partial ?_j}{\partial a_{kl}}\right) z_j(0) \quad (14)$$

If we are interested in how changes in one link affect all state variables, we can write:

$$\begin{bmatrix} \frac{\partial x_1(t)}{\partial a_{kl}} \\ \dots \\ \frac{\partial x_n(t)}{\partial a_{kl}} \end{bmatrix} = \begin{bmatrix} \left(\frac{\partial r_{11}}{\partial a_{kl}} e^{?_1 t} + r_{11}\frac{\partial e^{?_1 t}}{\partial ?_1}\frac{\partial ?_1}{\partial a_{kl}}\right) & \dots & \left(\frac{\partial r_{n1}}{\partial a_{kl}} e^{?_n t} + r_{n1}\frac{\partial e^{?_n t}}{\partial ?_n}\frac{\partial ?_n}{\partial a_{kl}}\right) \\ \dots & \dots & \dots \\ \left(\frac{\partial r_{1n}}{\partial a_{kl}} e^{?_1 t} + r_{1n}\frac{\partial e^{?_1 t}}{\partial ?_1}\frac{\partial ?_1}{\partial a_{kl}}\right) & \dots & \left(\frac{\partial r_{nn}}{\partial a_{kl}} e^{?_n t} + r_{nn}\frac{\partial e^{?_n t}}{\partial ?_n}\frac{\partial ?_n}{\partial a_{kl}}\right) \end{bmatrix} \begin{bmatrix} z_1(0) \\ \dots \\ z_n(0) \end{bmatrix} \quad (15)$$

Because the eigenvalues and eigenvectors in liner systems are constant, the derivative of the exponential of the *i*-th eigenvalue (e^{lit}) with respect to its eigenvalue (l_i) yield a term that depends on time (te^{lit}). Therefore, we can rewrite equation (15) to yield:

$$\begin{bmatrix} \frac{\partial x_1(t)}{\partial a_{kl}} \\ \dots \\ \frac{\partial x_n(t)}{\partial a_{kl}} \end{bmatrix} = \begin{bmatrix} \left(\frac{\partial r_{11}}{\partial a_{kl}} + r_{11}\frac{\partial ?_1}{\partial a_{kl}} t\right) e^{?_1 t} & \dots & \left(\frac{\partial r_{n1}}{\partial a_{kl}} + r_{n1}\frac{\partial ?_n}{\partial a_{kl}} t\right) e^{?_n t} \\ \dots & \dots & \dots \\ \left(\frac{\partial r_{1n}}{\partial a_{kl}} + r_{1n}\frac{\partial ?_1}{\partial a_{kl}} t\right) e^{?_1 t} & \dots & \left(\frac{\partial r_{nn}}{\partial a_{kl}} + r_{nn}\frac{\partial ?_n}{\partial a_{kl}} t\right) e^{?_n t} \end{bmatrix} \begin{bmatrix} z_1(0) \\ \dots \\ z_n(0) \end{bmatrix} \quad (16)$$

Equation (16) suggests that for each component j (with $j = 1$ to n) characterizing the behavior of state $x_i(t)$, the contribution to the change in behavior of state $x_i(t)$ due to the change in link gain (a_{kl}) is composed of two terms corresponding to:

1. The contribution of the derivative of r_{ji}, the *i*-th component of the *j*-th eigenvector, with respect to link gain (a_{kl}); and

[3] Note that the computation of the partial derivative of each term $r_{ji}e^{?_j t}z_j(0)$ assumes that the initial state $z_j(0)$ does not depend on the link gain. State $z_j(0)$ is a new state variable – obtained after the change of variables – given by ($\mathbf{z}(0) = \mathbf{V}^{-1}\mathbf{x}(0)$) where $\mathbf{z}(0)$ is the initial position vector of the new state variables and $\mathbf{x}(0)$ is the initial position vector of the original state variables. The inverse of the matrix of eigenvectors ($\mathbf{V}^{-1}$) depends on the value of all eigenvectors and thus varies with changes in the link gain. However, we do not differentiate it with respect to the loop gains because we can simply interpret it as a change in the initial position.

2. The contribution of the product of the r_{ji}, the *i*-th component of the *j*-th eigenvector, the derivative of the *i*-th eigenvalue (l_i) with respect to link gain (a_{kl}), and time (t).

The first term captures a change in intensity in the mode of behavior due to the contribution of the partial derivative of the the *i*-th component of the *j*-th eigenvector with respect to link gain (a_{kl}). Analogously, the second term captures a change in intensity in the mode of behavior, but it is more complicated. Here, the change in intensity grows with time, the *i*-th component of the *j*-th eigenvector and the partial derivative of the *i*-th eigenvalue (l_i) with respect to link gain (a_{kl}). Note that, if eigenvalues (l) and eigenvectors (**r**) are complex their derivatives will also be complex. In such cases, the exponentials will be multiplied by complex values which will influence not only the amplitude of the behavior mode, but will also lead to a phase shift.[4]

The equation above suggests that early in time ($t \cong 0$), the behavior mode will be mainly influenced by the first term, i.e., the derivative of the eigenvector with respect to the link gain; and later on (as $t \to \infty$), the behavior mode will be more influenced by the second term, i.e., the derivative of the eigenvalue with respect to the link gain. Therefore, the behavior of a linear system will be highly determined by the second component at a high value of t and the dominant mode of behavior will be determined by the relative size of each $r_{ji}\frac{\partial ?_j}{\partial a_{kl}}$. Since the majority of the research in model analysis has dealt with eigenvalue elasticity – closely associated with the derivative of the eigenvalue with respect to a link (or loop) – we have focused myopically at long term behavior impact, that is, how changes in links (or loops) affect the long term behavior of a state variable. These results may not play a significant role in the short term behavior of states of linear systems. This research can help differentiate the contribution of both eigenvectors and eigenvalues to the overall behavior of a state due to changes in link or loop gains.

3.1. Interpreting the Impact on Behavior Modes

To understand and interpret the impact that a change in link gains has on the original behavior of a state variable, it is useful to consider the ratio of the behavior of that state after a change in the link gain to the original one. Since each state is given by a linear combination of different behavior modes, we must also investigate the impact of the link change in each behavior mode component. The real part of the ratio (of changed state behavior to original one) determines a factor that multiplies the original behavior mode, either amplifying or dampening it. The complex part determines a phase gain to the original behavior mode. To obtain the behavior mode impact, we must divide each component in equation (14) by the corresponding component in equation (10):

$$\frac{\partial x_{ij}(t)/\partial a_{kl}}{x_{ij}(t)} = \frac{\left(\frac{\partial r_{ji}}{\partial a_{kl}} + r_{ji}\frac{\partial ?_j}{\partial a_{kl}}t\right)e^{?_j t} z_j(0)}{r_{ji}e^{?_j t} z_j(0)} = \frac{1}{r_{ji}}\frac{\partial r_{ji}}{\partial a_{kl}} + \frac{\partial ?_j}{\partial a_{kl}}t \qquad (17)$$

Equation (17) reemphasizes the role that the first time derivatives of both eigenvector and eigenvalue with respect to the link gain have on the new behavior of state $x_i(t)$. Since the ultimate goal of this formal model analysis is inform policy, it is important to compute the overall impact of changes by a link (or loop) gain to the overall behavior of a state. This overall impact requires addition of the individual impacts of different modes. Since the behavior modes are composed by a mix of oscillatory modes, exponential growth and decay and the coefficients change with time an automated implementation of the method will provide a mechanism to easily visualize the result and select the links or loops to change to obtain the desired behavior.

3.2. System Behavior: Link Eigenvalue and Link Eigenvector Sensitivities

Returning to equation (16), we observe that the partial derivatives of the eigenvalue (l_i) and eigenvector (r_{ji}) with respect to a link gain (a_{kl}), respectively $\frac{\partial ?_j}{\partial a_{kl}}$ and $\frac{\partial r_{ji}}{\partial a_{kl}}$, can be understood in the context of previous work on link gain eigenvalue elasticity (Forrester 1982, 1983). In his research Nathan Forrester (1982, 1983) suggested measuring the sensitivity of an eigenvalue with respect to a specific link (a_{kl}) by simply computing the partial derivative of the eigenvalue with respect to that link gain (a_{kl}). This would allow one to understand how the strength of a link could impact specific modes of behavior.

$$S_{l_i kl} = \frac{\P l_i}{\P a_{kl}} \qquad (18)$$

In addition, we could normalize the sensitivity measure to isolate the effect of the change in link gain from the magnitude of the eigenvalue and link gain. This normalization could be

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[4] See derivation in appendix A.

obtained multiplying the sensitivity by the ratio of the magnitude of the link gain (a_{kl}) to the magnitude of the eigenvalue (λ_i). He defined this measure *eigenvalue elasticity with respect to link gain* or *link gain (eigenvalue) elasticity.*

$$E_{ikl} = \frac{\partial \lambda_i}{\partial a_{kl}} \frac{|a_{kl}|}{\|\lambda_i\|} \qquad (19)$$

where $|a_{kl}|$ is the absolute value of the link gain and $\|\lambda_i\|$ is the Euclidean norm of a potentially complex eigenvalue (λ_i). Note that the partial derivative of the eigenvalue (λ_i) with respect to that link gain (a_{kl}) is present in the second term of equation (16) characterizing how a change in a link gain would affect the overall behavior of state $x_i(t)$.

While it has been suggested that eigenvector elasticity would be required to understand how structure ultimately influences behavior, no previous research other than ours and Saleh et al. (2005) has implemented it. To do so, define the eigenvector elasticity (r_{ji}) with respect to a link gain (a_{kl}) in a similar way as the link gain eigenvalue elasticity. First, we can measure the sensitivity of an eigenvector component (r_{ji}), the *i*-th component of the *j*-th eigenvector, with respect to a specific link (a_{kl}) by simply computing the partial derivative of the eigenvector component (r_{ji}) with respect to that link gain (a_{kl}), allowing one to understand how the strength of a link gain impacts the intensity of the eigenvector component.

$$S_{r_{ij}kl} = \frac{\partial r_{ij}}{\partial a_{kl}} \qquad (20)$$

Second, we could normalize the eigenvector sensitivity measure to isolate the effect of the change in link gain from the magnitude of the eigenvector component and link gain. This normalization could be obtained by multiplying the sensitivity by the ratio of the magnitude of the link gain (a_{kl}) to the magnitude of the eigenvector (r_{ij}). Third, instead of considering a specific eigenvector component we can account for the whole eigenvector ($\boldsymbol{r_j}$) and define this measure as the *eigenvector elasticity with respect to link gain* or *link gain eigenvector elasticity.*

$$E_{\mathbf{r_j}kl} = \frac{\partial \, \mathbf{r_j}}{\partial \, a_{kl}} \frac{|a_{kl}|}{\|\mathbf{r_j}\|} \qquad (21)$$

where $|a_{kl}|$ is the absolute value of the link gain and $\|\boldsymbol{r_j}\|$ is the Euclidean norm of the eigenvector ($\boldsymbol{r_j}$). Note that the partial derivative of the *i*-th component of the *j*-th eingevector (r_{ij})

with respect to the link gain (a_{kl}) is present in the first term of equation (16) characterizing how a change in a link gain affects the intensity of in the mode of behavior of eigenvalue i (l_i).

While the notion of link gain eigenvalue and eigenvector elasticities are useful, note that equation (16) provides an integrated way to assess how eigenvalue and eigenvector sensitivities (i.e., the partial derivatives with respect to a link gain) work together to influence system behavior. Rewriting equation (16) using eigenvalue and eigenvector sensitivities, we obtain:

$$\frac{\partial x_i(t)}{\partial a_{kl}} = \sum_{j=1}^{n} \left(S_{r_{ij}kl} + r_{ji} S_{l_j kl}\, t \right) e^{?_j t} z_j(0) \qquad (22)$$

- Eigenvector sensitivity $S_{r_{ij}kl} = \frac{\partial r_{ji}}{\partial a_{kl}}$ captures a change in intensity in the mode of behavior ($e^{?_j t} z_j(0)$) due to a change in a link gain (a_{kl})
- Eigenvalue sensitivity $S_{l_j kl} = \frac{\partial ?_j}{\partial a_{kl}}$ captures the change in the behavior mode (i.e., l_i) due to a change in the link gain (a_{kl}).
- The contribution of the eigenvalue sensitivity changes with time and it becomes the main determinant of behavior as time increases.

4. Behavior in Nonlinear Dynamic Systems

It is important to mention that the method of analysis as described so far applies only to linear systems, representing a very small subset of typical system dynamic models. Traditional system dynamics models are nonlinear, with eigenvalues and eigenvectors varying with time. Assuming that we could find a solution to the state vector **x**(t), the first time derivative of a nonlinear system (represented in the description of the method by equation 2) would also include the derivatives of the eigenvectors, leading to:

$$\dot{\mathbf{x}}(t) = [\dot{z}_1(t)\mathbf{r_1}(t) + z_1(t)\dot{\mathbf{r}}_\mathbf{1}(t)] + [\dot{z}_2(t)\mathbf{r_2}(t) + z_2(t)\dot{\mathbf{r}}_\mathbf{2}(t)] + \ldots + [\dot{z}_n(t)\mathbf{r_n}(t) + z_n(t)\dot{\mathbf{r}}_\mathbf{n}(t)] \qquad (23)$$

Note that the equation above is much more complicated that equation (4). Since the eigenvectors are linearly independent they span the n-dimensional space and we could write each derivative of an eigenvector $\dot{\mathbf{r}}_\mathbf{i}(t)$ as the linear combination of its projections on different eigenvectors. However, this prevents us from getting the desired separable state result of equation (6). Therefore, when we consider a nonlinear system the analysis becomes much more complicated.

Despite these complications, a possible way to still use the methodology derived above is to linearize the nonlinear system of equations. Since the linearized solutions are a good approximation of nonlinear systems solutions close to the operating point, the insights obtained locally (through linearization) cannot be generalized to the rest of the system. Nevertheless, we can circumvent this shortcoming by linearizing the system at every point in time (in practice, every time step in the simulation) and computing its eigenvalues and eigenvectors. Applying the methodology to the linearized system at every point in time allows us to compute how a change in link gains influence a change in the behavior of interest. Equation (22) provides a compact way to represent how changes in a link affect a state variable for a linear system, for a linearized system we could write a similar solution:

$$\frac{\partial x_i(t)}{\partial a_{kl}} = \sum_{j=1}^{n} \left(S_{r_{ij}kl} + r_{ji} S_{l_j kl} t \right) e^{?_j (t - t_0)} z_j(t_0) \qquad (24)$$

where each $z_j(t_0)$ refers to the position of the system at the linearization time (t_0).

Since the linearized system provides a good approximation to the nonlinear system only close to the operating point, we only care about solutions to equation (24) that happen early in time ($t \cong t_0$). The result of equation (24) at later times ($t \to \infty$) departs too far from where the system is a close approximation to the nonlinear system. Hence, for nonlinear systems that are linearized at every point in time, the impact of a change in link gain on system behavior can be simplified by substituting $t \cong t_0$ in equation (24). Equation (25) provides a good approximation of the impact of a change in link gains to the behavior of state x_i.

$$\frac{\partial x_i(t_0)}{\partial a_{kl}} = \sum_{j=1}^{n} \left(S_{r_{ij}kl} + r_{ji} S_{l_j kl} t_0 \right) z_j(t_0) \qquad (25)$$

Despite the additional complexity of nonlinear systems, by linearizing the system at every point in time and then considering the impact of the link gains, we arrive at a general solution that is similar to that of a linear system, with exception to the exponential multiplier. Equation (25) suggests that *eigenvector* sensitivity also plays an important role in determining the impact that a change in structure has on model behavior in nonlinear systems and it provides a framework to include it in the research in model analysis. We hope that follow up research implementing this method to nonlinear systems can shed more light on its usefulness to traditional system dynamics models.

5. Application to a Linear System: The Inventory-Workforce Oscillator

We illustrate the concepts above with a version of the familiar workforce inventory model. The model captures a simple production system. The model attempts to maintain desired inventory by adjusting production via hiring and firing workers. More precisely: Inventory integrates the difference between production and shipments. Shipments are determined by demand reduced by stock-outs, should inventory fall too low. Production depends on the workforce. And the workforce is "anchored" to the level necessary to meet expected demand. The workforce is increased above this anchor if inventory is too low and conversely workforce is decreased below the anchor if inventory is too high. Expected demand is a smooth of actual demand.

A stock and flow diagram of the model is shown below. The model is composed of three state variables, four flows, three auxiliary variables, two exogenous variables, and five constants.

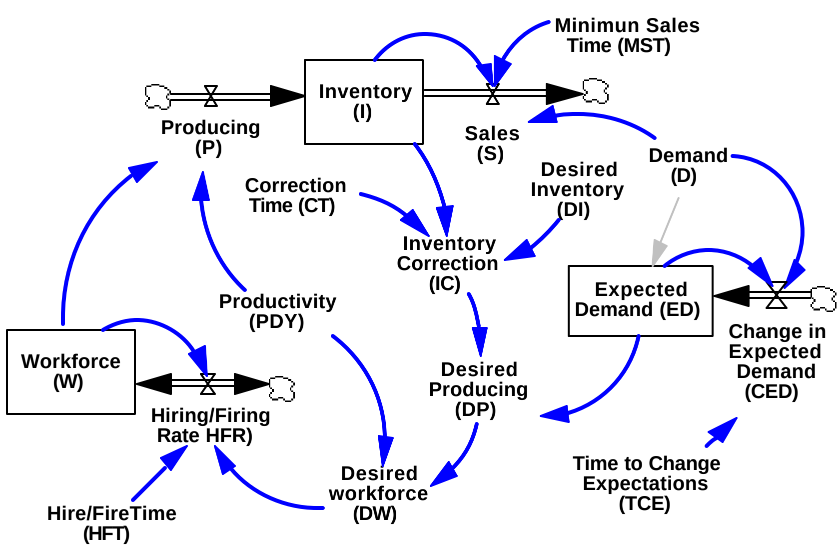

Figure 1 – Diagram of a linear system dynamics model.

$$\dot{I} = P - S = PDY \cdot W - D$$

$$\dot{W} = HFR = (DW - W)/HFT$$

$$\dot{ED} = CED = (D - ED)/TCE$$

$$IC = (DI - I)/CT$$

$$DP = IC + ED$$

$$DW = DP/PDY$$

The Jacobian (**J**) of the system above leads to the following relation:

$$J = \begin{bmatrix} 0 & PDY & 0 \\ -1/HFT \cdot PDY \cdot CT & -1/HFT & 1/HFT \cdot PDY \\ 0 & 0 & -1/TCE \end{bmatrix}$$

The results above represent the characteristic polynomial and the eigenvalues in terms of link gains. Analogously, we could have written the characteristic polynomial and eigenvalues in terms of loop gains. Since this system has only three loops:

Loop 1. A minor balancing loop associated with Workforce (*W*) with $g_1 = -1/HFT$.

Loop 2. A minor balancing loop associated with Expected Demand (*ED*), $g_2 = -1/TCE$.

Loop 3. A major balancing loop linking Inventory (I), Workforce (*W*), $g_3 = -1/(CT^* HFT)$.

it is straight forward to see that the characteristic polynomial reduces to:[5]

$$J = \begin{bmatrix} 0 & PDY & 0 \\ g_3/PDY & g_1 & -g_1/PDY \\ 0 & 0 & g_2 \end{bmatrix}$$

$$|l\,\mathrm{I} - \mathrm{J}| = \begin{bmatrix} l & -PDY & 0 \\ -g_3/PDY & l - g_1 & g_1/PDY \\ 0 & 0 & l - g_2 \end{bmatrix}$$

$$P(l) = l(l - g_1)(l - g_2) - g_3(l - g_2)$$

$$P(l) = l^3 + (-g_1 - g_2)\,l^2 + (g_1 g_2 - g_3)\,l + g_2 g_3$$

And, the eigenvalues, for the example, in terms of the loop gains are:

$$l_1 = g_2$$

$$l_2 = \frac{g_1}{2} - \frac{1}{2}\sqrt{g_1^{\,2} + 4g_3}$$

$$l_3 = \frac{g_1}{2} + \frac{1}{2}\sqrt{g_1^{\,2} + 4g_3}$$

We can easily compute the eigenvectors of the system using either link or loop gains, let us proceed with loop gains. The eigenvectors are given by:

($\boldsymbol{J r_i} = l_i \boldsymbol{r_i}$),

---

[5] The interested reader can also verify the derivation of the characteristic polynomial in terms of the loop gains in Gonçalves, Hines , Lertpattarapong (2000)

$$\begin{bmatrix} 0 & PDY & 0 \\ g_3/PDY & g_1 & -g_1/PDY \\ 0 & 0 & g_2 \end{bmatrix}\begin{bmatrix} r_{11} \\ r_{12} \\ r_{13} \end{bmatrix} = g_2 \begin{bmatrix} r_{11} \\ r_{12} \\ r_{13} \end{bmatrix}$$

$$PDYr_{12} = g_2 r_{11}$$

$$\frac{g_3}{PDY} r_{11} + g_1 r_{12} - \frac{g_1}{PDY} r_{13} = g_2 r_{12}$$

$$g_2 r_{13} = g_2 r_{13}$$

$$r_{13} = 1; r_{12} = \frac{g_2}{PDY} r_{11}; \frac{g_3 + g_1 g_2 - g_2^2}{PDY} r_{11} = \frac{g_1}{PDY}$$

$$\mathbf{r_1} = \begin{bmatrix} \dfrac{g_1}{(g_1 - g_2)g_2 + g_3} \\ \dfrac{g_1 g_2}{((g_1 - g_2)g_2 + g_3)PDY} \\ 1 \end{bmatrix}; \mathbf{r_2} = \begin{bmatrix} -\dfrac{\left(g_1 + \sqrt{g_1^2 + 4g_3}\right)PDY}{2g_3} \\ 1 \\ 0 \end{bmatrix}; \mathbf{r_3} = \begin{bmatrix} \dfrac{\left(-g_1 + \sqrt{g_1^2 + 4g_3}\right)PDY}{2g_3} \\ 1 \\ 0 \end{bmatrix}$$

We can then represent the system behavior in matrix form:

$$\begin{bmatrix} I(t) \\ W(t) \\ ED(t) \end{bmatrix} = \begin{bmatrix} \dfrac{g_1}{(g_1 - g_2)g_2 + g_3} & -\dfrac{\left(g_1 + \sqrt{g_1^2 + 4g_3}\right)PDY}{2g_3} & \dfrac{\left(-g_1 + \sqrt{g_1^2 + 4g_3}\right)PDY}{2g_3} \\ \dfrac{g_1 g_2}{((g_1 - g_2)g_2 + g_3)PDY} & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{g_2 t} z_1(0) \\ e^{\frac{1}{2}\left(g_1 - \sqrt{g_1^2 + 4g_3}\right)t} z_2(0) \\ e^{\frac{1}{2}\left(g_1 + \sqrt{g_1^2 + 4g_3}\right)t} z_3(0) \end{bmatrix}$$

Expanding the equations above, we obtain the system below:

$$I(t) = \frac{g_1}{(g_1 - g_2)g_2 + g_3} e^{g_2 t} z_1(0) - \frac{\left(g_1 + \sqrt{g_1^2 + 4g_3}\right)PDY}{2g_3} e^{\frac{1}{2}\left(g_1 - \sqrt{g_1^2 + 4g_3}\right)t} z_2(0) + \frac{\left(-g_1 + \sqrt{g_1^2 + 4g_3}\right)PDY}{2g_3} e^{\frac{1}{2}\left(g_1 + \sqrt{g_1^2 + 4g_3}\right)t} z_3(0)$$

$$W(t) = \frac{g_1 g_2}{((g_1 - g_2)g_2 + g_3)PDY} e^{g_2 t} z_1(0) + e^{\frac{1}{2}\left(g_1 - \sqrt{g_1^2 + 4g_3}\right)t} z_2(0) + e^{\frac{1}{2}\left(g_1 + \sqrt{g_1^2 + 4g_3}\right)t} z_3(0)$$

$$ED(t) = e^{g_2 t} z_1(0)$$

The system of equations above permits us to compute the dominant behavior modes by comparing the eigenvector components for each behavior mode that influence a state. With this purpose, we allow the time constants for inventory correction time (*CT*), hire-fire time (*HFT*), and change demand expectations (*TCE*) to equal (e.g. 2 months), we obtain that $g_1 = -1/HFT = -1/2$, $g_2 = -1/TCE = -1/2$, $g_3 = -1/(CT^* HFT) = -1/4$, and $PDY = 10$, providing us with the following eigenvectors:

$$\mathbf{r_1} = \begin{bmatrix} 2 \\ 0.1 \\ 1 \end{bmatrix}; \mathbf{r_2} = \begin{bmatrix} -10 + i10\sqrt{3} \\ 1 \\ 0 \end{bmatrix}; \mathbf{r_3} = \begin{bmatrix} -10 - i10\sqrt{3} \\ 1 \\ 0 \end{bmatrix}$$

Substituting them in the equations describing the behavior of state variables

$$I(t) = 2e^{-0.5t} z_1(0) + \left(-10 + i10\sqrt{3}\right) e^{-\frac{1}{4}\left(1+i\sqrt{3}\right)t} z_2(0) + \left(-10 - i10\sqrt{3}\right) e^{-\frac{1}{4}\left(1-i\sqrt{3}\right)t} z_3(0)$$

$$W(t) = 0.1e^{-0.5t} z_1(0) + e^{-\frac{1}{4}\left(1+i\sqrt{3}\right)t} z_2(0) + e^{-\frac{1}{4}\left(1-i\sqrt{3}\right)t} z_3(0)$$

$$ED(t) = e^{-0.5t} z_1(0)$$

Or in matrix form: $\begin{bmatrix} I(t) \\ W(t) \\ ED(t) \end{bmatrix} = \begin{bmatrix} 2 & -10\left(1-i\sqrt{3}\right) & -10\left(1+i\sqrt{3}\right) \\ -0.1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{-\frac{1}{2}t} z_1(0) \\ e^{-\frac{1}{4}\left(1+i\sqrt{3}\right)t} z_2(0) \\ e^{-\frac{1}{4}\left(1-i\sqrt{3}\right)t} z_3(0) \end{bmatrix}$

The dominant behavior of state $ED(t)$ is the exponential decay with rate $g_2(= -0.5)$. Comparing the magnitudes of the coefficients of the exponential terms in $I(t)$ and $W(t)$, we observe that the dominant behavior of those states is a decaying exponential, determined by the pair of complex eigenvalues. Note also that in this simple system, only loop gains 1 (g_1) and 3 (g_3) influence the dominant behavior of $I(t)$ and $W(t)$; and only loop gain 2 (g_2) influences the behavior of $ED(t)$. To understand how the state variables are impacted by changes in loop (or link) gains, we need to compute both the derivatives of eigenvalues and eigenvectors with respect to the loop (or link) gains. In the derivation that follows we use loop gains. Equation (31) provides a framework to integrate these impacts and tables 1 and 2 presents the necessary derivatives.

Table 1 – Derivatives of eigenvalues wrt loop gains for inventory-workforce example.

	Eigenvalue 1 $l_1 = g_2$	Eigenvalue 2 $l_2 = \frac{g_1}{2} - \frac{1}{2}\sqrt{g_1^2 + 4g_3}$	Eigenvalue 3 $l_3 = \frac{g_1}{2} + \frac{1}{2}\sqrt{g_1^2 + 4g_3}$
Loop 1 Hiring (g_1)	$\frac{\partial l_1}{\partial g_1} = 0$	$\frac{\partial l_2}{\partial g_1} = \frac{1}{2}\left(1 - \frac{g_1}{\sqrt{g_1^2 + 4g_3}}\right)$	$\frac{\partial l_3}{\partial g_1} = \frac{1}{2}\left(1 + \frac{g_1}{\sqrt{g_1^2 + 4g_3}}\right)$
Loop 2 Demand Adj. (g_2)	$\frac{\partial l_1}{\partial g_2} = 1$	$\frac{\partial l_2}{\partial g_2} = 0$	$\frac{\partial l_3}{\partial g_2} = 0$
Loop 3 Inventory-wkforce (g_3)	$\frac{\partial l_1}{\partial g_3} = 0$	$\frac{\partial l_2}{\partial g_3} = -\frac{1}{\sqrt{g_1^2 + 4g_3}}$	$\frac{\partial l_3}{\partial g_3} = +\frac{1}{\sqrt{g_1^2 + 4g_3}}$

First, note that the derivative of the eigenvalue 2 and 3 are not influenced by loop gain 2 (the derivatives are equal to zero.) Second, loop 3 does not affect the dampening of the complex eigenvalues. In addition, note that increasing g_1 decreases the frequency (increases the period) of oscillation. The complex part in the derivative has a different sign than the sign of the eigenvalue's complex part (b). Therefore**, a change in g_1 decreases the complex part of the eigenvalue** and since f= 2πb (or T = 2π/b) a lower value of b leads to slower frequency (or, a longer period.) Analogously, increasing g_3 increases the frequency of oscillation, since the complex part of the derivative has the same sign as the sign of the eigenvalue's complex part (b).

Table 2 – Derivatives of eigenvectors wrt loop gains for inventory-workforce example.

	Eigenvector 1 $\mathbf{r_1} = \left[\frac{g_1}{(g_1 - g_2)g_2 + g_3} \quad \frac{g_1 g_2}{((g_1 - g_2)g_2 + g_3)PDY} \quad 1\right]$	Eigenvector 2 $\mathbf{r_2} = \left[-\frac{\left(g_1 + \sqrt{g_1^2 + 4g_3}\right)PDY}{2g_3} \quad 1 \quad 0\right]$	Eigenvector 3 $\mathbf{r_3} = \left[\frac{\left(-g_1 + \sqrt{g_1^2 + 4g_3}\right)PDY}{2g_3} \quad 1 \quad 0\right]$
Loop 1 Hiring (g_1)	$\frac{\partial \mathbf{r_1}}{\partial g_1} = \left[\frac{-g_2^2 + g_3}{((g_1 - g_2)g_2 + g_3)^2} \quad \frac{\left(-g_2^2 + g_3\right)g_2}{((g_1 - g_2)g_2 + g_3)^2 PDY} \quad 0\right]$	$\frac{\partial \mathbf{r_2}}{\partial g_1} = \left[-\frac{PDY}{2g_3}\left(1 + \frac{g_1}{\sqrt{g_1^2 + 4g_3}}\right) \quad 0 \quad 0\right]$	$\frac{\partial \mathbf{r_3}}{\partial g_1} = \left[\frac{PDY}{2g_3}\left(-1 + \frac{g_1}{\sqrt{g_1^2 + 4g_3}}\right) \quad 0 \quad 0\right]$
Loop 2 Demand Adj. (g_2)	$\frac{\partial \mathbf{r_1}}{\partial g_2} = \left[\frac{-g_1(g_1 - 2g_2)}{((g_1 - g_2)g_2 + g_3)^2} \quad \frac{g_1\left(g_2^2 + g_3\right)}{((g_1 - g_2)g_2 + g_3)^2 PDY} \quad 0\right]$	$\frac{\partial \mathbf{r_2}}{\partial g_2} = [0 \quad 0 \quad 0]$	$\frac{\partial \mathbf{r_3}}{\partial g_2} = [0 \quad 0 \quad 0]$
Loop 3 Inventory- wkforce (g_3)	$\frac{\partial \mathbf{r_1}}{\partial g_3} = \left[\frac{-g_1}{((g_1 - g_2)g_2 + g_3)^2} \quad \frac{-g_1 g_2}{((g_1 - g_2)g_2 + g_3)^2 PDY} \quad 0\right]$	$\frac{\partial \mathbf{r_2}}{dg_3} = \left[\frac{PDY}{2g_3^2}\left(g_1 + \frac{g_1^2 + 2g_3}{\sqrt{g_1^2 + 4g_3}}\right) \quad 0 \quad 0\right]$	$\frac{\partial \mathbf{r_3}}{dg_3} = \left[\frac{PDY}{2g_3^2}\left(g_1 - \frac{g_1^2 + 2g_3}{\sqrt{g_1^2 + 4g_3}}\right) \quad 0 \quad 0\right]$

Before we proceed, we should consider the impact of the changes of loop gains in the eigenvectors. Focusing mainly on the oscillatory eigenvalues let us consider the derivative of r_{21} with respect to g_1. First, the real part suggests that every incremental change in g_1 causes a multiplication of (-PDY/g_3). The complex part of the derivative suggests a reduction in the complex value b, reducing the phase lag that it could have on the system behavior. Since the real and complex parts have the same sign the phase lag is positive. Loop 3 has a similar impact on the phase lag. Incorporating the results from tables 1 and 2 in equation (21) provides an integrated way to assess how the partial derivatives of the states with respect to a loop gain influence system behavior.

$$\begin{bmatrix} \frac{\partial I(t)}{\partial g_1} \\ \frac{\partial W(t)}{\partial g_1} \\ \frac{\partial ED(t)}{\partial g_1} \end{bmatrix} = \begin{bmatrix} \frac{-g_2^2+g_3}{((g_1-g_2)g_2+g_3)^2} & -\frac{PDY}{2g_3\sqrt{g_1^2+4g_3}}\left(g_1+\sqrt{g_1^2+4g_3}+2g_3t\right) & \frac{PDY}{2g_3\sqrt{g_1^2+4g_3}}\left(g_1-\sqrt{g_1^2+4g_3}+2g_3t\right) \\ \frac{(-g_2^2+g_3)g_2}{((g_1-g_2)g_2+g_3)^2 PDY} & \frac{1}{2}\left(1-\frac{g_1}{\sqrt{g_1^2+4g_3}}\right)t & \frac{1}{2}\left(1+\frac{g_1}{\sqrt{g_1^2+4g_3}}\right)t \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{g_2 t} z_1(0) \\ e^{\frac{1}{2}\left(g_1-\sqrt{g_1^2+4g_3}\right)t} z_2(0) \\ e^{\frac{1}{2}\left(g_1+\sqrt{g_1^2+4g_3}\right)t} z_3(0) \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial I(t)}{\partial g_2} \\ \frac{\partial W(t)}{\partial g_2} \\ \frac{\partial ED(t)}{\partial g_2} \end{bmatrix} = \begin{bmatrix} \left(\frac{-g_1(g_1-2g_2)}{((g_1-g_2)g_2+g_3)^2}+\frac{g_1}{(g_1-g_2)g_2+g_3}t\right) & 0 & 0 \\ \left(\frac{g_1(g_2^2+g_3)}{((g_1-g_2)g_2+g_3)^2 PDY}+\frac{g_1 g_2}{((g_1-g_2)g_2+g_3)PDY}t\right) & 0 & 0 \\ t & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{g_2 t} z_1(0) \\ e^{\frac{1}{2}\left(g_1-\sqrt{g_1^2+4g_3}\right)t} z_2(0) \\ e^{\frac{1}{2}\left(g_1+\sqrt{g_1^2+4g_3}\right)t} z_3(0) \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial I(t)}{\partial g_3} \\ \frac{\partial W(t)}{\partial g_3} \\ \frac{\partial ED(t)}{\partial g_3} \end{bmatrix} = \begin{bmatrix} \frac{-g_1}{((g_1-g_2)g_2+g_3)^2} & \frac{PDY}{2g_3}\left[\left(\frac{g_1}{g_3}+\frac{g_1^2+2g_3}{g_3\sqrt{g_1^2+4g_3}}\right)+\left(1+\frac{g_1}{\sqrt{g_1^2+4g_3}}\right)t\right] & \frac{PDY}{2g_3}\left[\left(\frac{g_1}{g_3}-\frac{g_1^2+2g_3}{g_3\sqrt{g_1^2+4g_3}}\right)+\left(1-\frac{g_1}{\sqrt{g_1^2+4g_3}}\right)t\right] \\ \frac{-g_1 g_2}{((g_1-g_2)g_2+g_3)^2 PDY} & -\frac{1}{\sqrt{g_1^2+4g_3}}t & \frac{1}{\sqrt{g_1^2+4g_3}}t \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{g_2 t} z_1(0) \\ e^{\frac{1}{2}\left(g_1-\sqrt{g_1^2+4g_3}\right)t} z_2(0) \\ e^{\frac{1}{2}\left(g_1+\sqrt{g_1^2+4g_3}\right)t} z_3(0) \end{bmatrix}$$

Each mode of behavior ($e^{?_j t}$) is multiplied by a (potentially complex) factor $\left(\frac{\partial r_{ji}}{\partial g_k}+r_{ji}\frac{\partial ?_j}{\partial g_k}t\right)$, influencing the intensity of the original behavior mode and potentially the phase lag. The results may be easier to interpret after we substitute values for each of the loop gains. Substituting the values for each loop gain (g_1, g_2 and g_3) and productivity (*PDY*) suggests that the oscillatory modes remain dominant.

$$\begin{bmatrix} \frac{\partial I(t)}{\partial g_1} \\ \frac{\partial W(t)}{\partial g_1} \\ \frac{\partial ED(t)}{\partial g_1} \end{bmatrix} = \begin{bmatrix} -8 & \left(20+i\frac{20\sqrt{3}}{3}\right)+i\left(\frac{20\sqrt{3}}{3}\right)t & \left(20-i\frac{20\sqrt{3}}{3}\right)-i\left(\frac{20\sqrt{3}}{3}\right)t \\ 0.4 & \frac{1}{2}\left(1-i\frac{\sqrt{3}}{3}\right)t & \frac{1}{2}\left(1+i\frac{\sqrt{3}}{3}\right)t \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{-\frac{1}{2}t} z_1(0) \\ e^{-\frac{1}{4}(1+i\sqrt{3})t} z_2(0) \\ e^{-\frac{1}{4}(1-i\sqrt{3})t} z_3(0) \end{bmatrix}; \quad \begin{bmatrix} \frac{\partial I(t)}{\partial g_2} \\ \frac{\partial W(t)}{\partial g_2} \\ \frac{\partial ED(t)}{\partial g_2} \end{bmatrix} = \begin{bmatrix} (4+2t) & 0 & 0 \\ (-0.1t) & 0 & 0 \\ t & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{-\frac{1}{2}t} z_1(0) \\ e^{-\frac{1}{4}(1+i\sqrt{3})t} z_2(0) \\ e^{-\frac{1}{4}(1-i\sqrt{3})t} z_3(0) \end{bmatrix};$$

$$\begin{bmatrix} \frac{\partial I(t)}{\partial g_3} \\ \frac{\partial W(t)}{\partial g_3} \\ \frac{\partial ED(t)}{\partial g_3} \end{bmatrix} = \begin{bmatrix} 8 & \left(-40+i\frac{40\sqrt{3}}{3}\right)-\left(20+i\frac{20\sqrt{3}}{3}\right)t & \left(-40-i\frac{40\sqrt{3}}{3}\right)-\left(20-i\frac{20\sqrt{3}}{3}\right)t \\ -0.4 & i\frac{2\sqrt{3}}{3}t & -i\frac{2\sqrt{3}}{3}t \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{-\frac{1}{2}t} z_1(0) \\ e^{-\frac{1}{4}\left(1+i\sqrt{3}\right)t} z_2(0) \\ e^{-\frac{1}{4}\left(1-i\sqrt{3}\right)t} z_3(0) \end{bmatrix}$$

We can make sense of the impact introduced by changes in the loop gains by comparing the cells of each of the three matrices above with cells in the original solution matrix (reproduced below), according to the result from equation (17).

$$\begin{bmatrix} I(t) \\ W(t) \\ ED(t) \end{bmatrix} = \begin{bmatrix} 2 & -10\left(1-i\sqrt{3}\right) & -10\left(1+i\sqrt{3}\right) \\ -0.1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{-\frac{1}{2}t} z_1(0) \\ e^{-\frac{1}{4}\left(1+i\sqrt{3}\right)t} z_2(0) \\ e^{-\frac{1}{4}\left(1-i\sqrt{3}\right)t} z_3(0) \end{bmatrix}$$

For instance, it is possible to see that a change in gain 2 (g_2) does not have an impact on the oscillatory mode of behavior. The result makes intuitive sense because loop 2, a minor balancing loop associated with Expected Demand (*ED*), does not contribute to the generation of the oscillatory mode, as can be seen from eigenvalues 2 and 3. Nevertheless, a change in g_2 impacts all states in the system, increasing the amplitude associated with the exponential decay. Note also that the size of the change is dependent on time, resulting from the amplification of the change over time due to the change in loop gain.

The equations above also suggest that changes in loop gain 1 (g_1) do not impact the behavior of expected demand (*ED*), which can be seen by a row of zeros in the respective gain matrices. Furthermore, the change in g_1 amplifies the original exponential decay ($e^{-\frac{1}{2}t}$) by a factor of four while also changing its sign. Perhaps more difficult to understand is the impact on the oscillatory mode of behavior, seen in the coefficients for both $e^{-\frac{1}{4}\left(1+i\sqrt{3}\right)t}$ and $e^{-\frac{1}{4}\left(1-i\sqrt{3}\right)t}$. Again, the real part of the ratio (of the changed state behavior to the original one) determines a factor that multiplies the original behavior mode; and the complex part determines a phase gain to the original behavior mode. Consider first the impact of a change in g_1 on inventory (*I*)'s behavior mode $e^{-\frac{1}{4}\left(1+i\sqrt{3}\right)t}$, the ratio between changed and original state results in $\frac{t}{2}-i\left(\frac{2\sqrt{3}}{3}+\frac{\sqrt{3}}{6}t\right)$. The result suggests that the impact depends on time. The complex coefficient contributes to the amplification with the square root of the sum of squares of the real and complex parts

($\sqrt{\left(\frac{t}{2}\right)^2+\left(-\frac{2\sqrt{3}}{3}-\frac{\sqrt{3}}{6}t\right)^2}$) and to the phase shift by the inverse tangent of the ratio of the real by the complex parts ($tan^{-1}\left(-\left(\frac{2\sqrt{3}}{3}+\frac{\sqrt{3}}{6}t\right)\Big/\left(\frac{t}{2}\right)\right)$). When time is close to zero ($t \cong 0$), the amplification to the oscillatory mode is given by a factor of $\frac{2\sqrt{3}}{3}$ and the phase shift is of $-\frac{p}{2}$. To compute the impact on the inventory (*I*) behavior at a specific time *t*, it would be required to substitute the adequate value of time. For instance, at $t = 4$ the change in g_1 causes an amplification to the oscillatory mode by a factor of 3.05 (since $\sqrt{\left(\frac{4}{2}\right)^2+\left(-\frac{4\sqrt{3}}{3}\right)^2}=\sqrt{\frac{28}{3}}=3.05$) and a phase shift of approximately -49^{o} (since $tan^{-1}\left(-2\sqrt{3}/3\right)\approx -49^{o}$). It is necessary to proceed in a similar way to compute the impact on different behavior modes. To inform policy it is still required to compute the overall impact of changes in a loop gain to the overall behavior of a state, by adding the individual impacts of different modes and selecting the desired behavior modes.

5. Discussion

The motivation for this paper is to provide a mathematical framework to understand the contribution that changes in link (or loop) gains have on the time path behavior of state variables in linear dynamic systems. Our research focuses on the analytical computation of the influence of eigenvalues and eigenvectors on model behavior. This work follows closely the research tradition established by Forrester (1982). Our work departs from previous efforts in terms of its focus on analytical results and emphasis on the impact that first time derivatives of eigenvalues and eigenvectors have on model behavior, instead of eigenvalue elasticities.

The method discussed above has the advantage of introducing an analytical understanding of the role of eigenvectors to influence behavior in linear systems; it is precise, it is reproducible; and it provides a standard way to analyze linear dynamic models. Second, the method provides a direct measure of the impact of different loops on the behavior response of the system. Third, the method characterizes measures and enumerates how different loops influence different modes of behavior. Fourth, the method contributes to understanding of transient analysis instead of simply steady state analysis. Finally, by linearizing a nonlinear system at every point in time, we arrive at a general solution that provides a good approximation of the impact of a change in link gains to the behavior of state x_i.

The method also has a number of shortcomings. First, solutions to the behavior of states in the system are required to obtain the analytical results. Also, the derivations aimed at the impact of a change in structure to the behavior of linear systems. While linear systems are used to derive the main results, consecutive system linearization extends the application to nonlinear systems. This result is stated but no example is provided. Further research that implements the computation of eigenvalues, eigenvectors and the first derivatives with respect to the link (and loop) gains and test different nonlinear models are required to assess the usefulness of the proposed method. As the linear example suggests, the method poses challenges in terms of interpreting and evaluating the impact of eigenvector and eigenvalue contribution to behavior modes.

Despite its current challenges and limitations, we are hopeful that the method provides a useful step to the analysis of how structure influences behavior as well as a new direction for future research on the analysis of nonlinear dynamic systems.

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Appendix A – The Product of a Complex Number by Complex Exponentials

To understand the implication of multiplying a complex exponential by a complex number, consider the following example:

$$(a+b\mathrm{i})e^{(c+d\mathrm{i})}$$

we can rewrite the exponential as:

$$e^{(c+d\mathrm{i})} = e^{c}e^{d\mathrm{i}}$$

and by definition

$$e^{d\mathrm{i}} = cos(d) + \mathrm{i}\, sin(d)$$

so we can rewrite the equation above as:

$$e^{c}(a+b\mathrm{i})(cos(d) + \mathrm{i}\, sin(d))$$

$$(ae^{c})(cos(d) + \mathrm{i}\, sin(d)) + (be^{c})(\mathrm{i}\, cos(d) - sin(d))$$

$$e^{c}[(a\, cos(d) - b\, sin(d)) + \mathrm{i}(b\, cos(d) + a\, sin(d))]$$

Multiplying by $1(\frac{\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}})$ and defining $\frac{b}{a} = tan(f)$, we observe that $\frac{a}{\sqrt{a^2+b^2}} = cos(f)$ and $\frac{b}{\sqrt{a^2+b^2}} = sin(f)$ we can rewrite the equation above as:

$$\left(\sqrt{a^2+b^2}\right)e^{c}\left[\left(\frac{a}{\sqrt{a^2+b^2}}cos(d) - \frac{b}{\sqrt{a^2+b^2}}sin(d)\right) + \mathrm{i}\left(\frac{b}{\sqrt{a^2+b^2}}cos(d) + \frac{a}{\sqrt{a^2+b^2}}sin(d)\right)\right]$$

$$\left(\sqrt{a^2+b^2}\right)e^{c}[(cos(f)cos(d) - sin(f)sin(d)) + \mathrm{i}(sin(f)cos(d) + cos(f)sin(d))]$$

Since $cos(d+f) = (cos(f)cos(d) - sin(f)sin(d))$ and $sin(d+f) = (sin(f)cos(d) + cos(f)sin(d))$, we obtain:

$$\left(\sqrt{a^2+b^2}\right) e^{c}[cos(d+f) + \mathrm{i}\, sin(d+f)]$$

$$\left(\sqrt{a^2+b^2}\right) e^{c+\mathrm{i}(d+f)}$$

$$\left(\sqrt{a^2+b^2}\right) e^{c+\mathrm{i}\left(d+tan^{-1}\left(\frac{b}{a}\right)\right)}$$

Therefore, the complex number multiplying the exponential contributes to the amplification with the square root of the sum of squares of the real and complex parts, and to the phase shift by the inverse tangent of the ratio of the complex by the real parts. The inverse tangent of (x) is defined in the interval $-\frac{p}{2} < f < \frac{p}{2}$. The inverse tangent takes a value of zero when x is zero; and it takes a positive (negative) value when x is positive (negative).

Appendix B - How loops influence system behavior?

To understand how changes in loop gains (i.e., the strength of a feedback loop) influence system behavior, we follow a derivation analogous to the one in section 3. The behavior of each state in the system $x_i(t)$ is described by equation (10), which demonstrates that the behavior of each state is influenced both by eigenvalues (l_i) and eigenvectors (r_{ji}).

$$x_i(t) = r_{1i} e^{l_1 t} z_1(0) + r_{2i} e^{l_2 t} z_2(0) + \ldots + r_{ni} e^{l_n t} z_n(0)$$

While it is more common to write the characteristic polynomial ($P(l)$) and eigenvalues in terms of the link gains (a_{kl}), it is also possible to write them in terms of loop gains (g_k). Loops, and their gains, may be a more comprehensive (better) way to describe structure, since modelers often decide to include (or exclude) loops based on the dynamic hypotheses that they believe are important in a system. Since we are ultimately interested in how structure drives behavior, understanding how changes in loop gains influence system behavior may be more appropriate than looking at how changes in links influence behavior. To capture how loops influence system behavior, we take the partial derivative of each state in the system $x_i(t)$ with respect to its loop gains. Therefore we take a partial derivative of equation (10), characterizing the behavior of state $x_i(t)$, with respect to a loop gain (g_k).

$$\frac{\partial x_i(t)}{\partial g_k} = \frac{\partial}{\partial g_k}\left[r_{1i} e^{?_1 t} z_1(0) + \ldots + r_{ni} e^{?_n t} z_n(0)\right] \quad \text{(B1)}$$

Which for linear systems, we can write equation as:

$$\frac{\partial x_i(t)}{\partial g_k} = \sum_{j=1}^{n} \left(\frac{\partial r_{ji}}{\partial g_k} + r_{ji} \frac{\partial ?_j}{\partial g_k} t \right) e^{?_j t} z_j(0) \quad \text{(B2)}$$

Equation (B2) suggests that for each component j (with $j = 1$ to n) characterizing the behavior of state $x_i(t)$, the contribution to the change in behavior of state $x_i(t)$ due to the change in loop gain (g_k) is composed of two terms. The first term captures a change in intensity in the mode of behavior due to the contribution of the partial derivative of the the i-th component of the j-th eigenvector with respect to loop gain (g_k). Analogously, the second term captures a change in intensity in the mode of behavior due to (a) time, (b) the i-th component of the j-th eigenvector and (c) the partial derivative of the i-th eigenvalue (l_i) with respect to loop gain (g_k).

With his suggestion of finding the characteristic polynomial in terms of the loop gains, Forrester (1983) extended the results of link sensitivity and link elasticity to loop sensitivity and loop elasticity.

$$S_{\lambda_i k} = \frac{\partial \lambda_i}{\partial g_k} \text{ and } E_{ik} = \frac{\partial \lambda_i}{\partial g_k}\frac{|g_k|}{\|\lambda_i\|} \qquad \text{(B3)}$$

In addition, we can extend the concept of link eigenvector sensitivity and elasticity introduced in the previous section to loop *eigenvector sensitivity* and eigenvector elasticity with respect to loop gain or *loop gain eigenvector elasticity.*

$$S_{r_{ij} k} = \frac{\partial r_{ij}}{\partial g_k} \text{ and } E_{\mathbf{r_j} k} = \frac{\partial \mathbf{r_j}}{\partial g_k}\frac{|g_k|}{\|\mathbf{r_j}\|} \qquad \text{(B4)}$$

Equation (B2) provides an integrated way to assess how loop eigenvalue and eigenvector sensitivity (i.e., the partial derivatives with respect to a loop gain) work together to influence system behavior. In particular, we can rewrite equation (B2) as:

$$\frac{\partial x_i(t)}{\partial g_k} = \sum_{j=1}^{n}\left(S_{r_{ij}k} + r_{ji} S_{\lambda_j k} t\right) e^{\lambda_j t} z_j(0) \qquad \text{(B5)}$$

- Loop eigenvector sensitivity $S_{r_{ij}k} = \frac{\partial r_{ij}}{\partial g_k}$ captures a change in intensity in the mode of behavior ($e^{\lambda_j t} z_j(0)$) due to a change in a loop gain (g_k);
- Loop eigenvalue sensitivity $S_{\lambda_i k} = \frac{\partial \lambda_i}{\partial g_k}$ captures the change in the behavior mode (i.e., λ_i) due to a change in the loop gain (g_k); and
- The contribution of the eigenvalue elasticity changes with time, becoming the main determinant of behavior over time.

Loop gain eigenvalue elasticity captures changes in the mode of behavior, that is, it measures whether the state will have faster or slower growth, decay, or oscillations. In turn, loop gain eigenvector elasticity capture changes in the intensity of that behavior mode, that is, it measures the importance of that behavior mode to the overall behavior of the state. To compute the eigenvalues in terms of loop gains readers are directed to Forrester (1983), Kampmann (1996), Gonçalves, Hines and Lertpattarapong (2000) and Kampmann and Oliva (2006).