Do the Parallel Lines Meet?
A Comparison between Pathway Participation Metrics and Eigenvalue Analysis

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Abstract:
The search for tools and techniques aimed at model analysis has gained momentum in the system dynamics community during the last decade. A variety of approaches have been developed, modified and applied to replace or enhance the traditional intuitive-based schemes for understanding the behavior of dynamic models according to its feedback structure. Despite the diversity of the methods developed for model analysis, recent studies suggest that there is considerable convergence in the results they produce. The objective of this paper is to explore some of the similarities and differences between pathway participation metrics and eigenvalue elasticity analysis that can potentially explain the reasons for the convergence and divergence in analysis. In the first part of the paper, we lay out some of the theoretical differences and similarities in approaches between aspects of pathway participation metrics and eigenvalue elasticity analysis in order to explain how they, despite the significant differences in the process of model analysis, can produce similar and comparable results under certain conditions. The second part of the paper presents application of pathway participation metrics in four well-known models that have been previously analyzed using the eigenvalue elasticity approach. The case studies highlight some of the similarities and differences between the two approaches in detecting the dominant structure.

INTRODUCTION

The search for tools and techniques aimed at model analysis has gained momentum in the system dynamics community during the last decade. A variety of approaches have been developed, modified and applied to replace or enhance the traditional intuitive-based schemes for understanding the behavior of dynamic models according to its feedback structure. The breadth and scope of the effort is mainly technical and is targeted towards a small interest group. However, the ultimate goal is to reach a wider audience that works with models as decision and policy support tools. The ambition is to equip the commercial system dynamics software packages with a toolbox for model analysis to facilitate "telling correct, cogent and coherent dynamic stories" (Mojtahedzadeh, 1996), to support "modeling at the speed of conversation" (Hines, 2003) and to facilitate the identification of leverage points (Forrester, 1982). Sterman (2000) suggests that "Understanding model behavior goes beyond the invocation of simple archetypes …While true, these statements don't provide the deep insight into model structure and behavior required to develop your intuition about dynamics or your ability to identify high leverage policies.” Although the battle is not yet won, we are continually getting closer to accomplishing the goal of reliable and consistent automated model analysis.

Despite the diversity of the methods developed for model analysis, there is a considerable convergence in the results they produce. A number of comparative studies conducted to contrast the outcome of alternative techniques for model analysis have concluded that a significant overlap may exist among the outcomes of various approaches to model analysis (Ford, 2005; Olivia, 2004; Kampmann, 2006; Güneralp, 2006). While the overlap may not be unexpected as all the outcomes are filtered through the intuition of the developer who interprets the results, an interesting theme for research is to look for the fundamental basis that can create convergence and divergence.

The objective of this paper is to explore some of the similarities and differences between pathway participation metrics and eigenvalue elasticities analysis that can potentially lead the convergence and divergence in the outcome of model analysis. Kampman and Olivia (2006) compare the results of eigenvalue elasticity analysis and pathway participation metrics in Industrial Structures model (Mojtahedzadeh et al, 2004) and conclude that the dominant structure identified by the two methods are “to a large extent, in accordance” with one another. Güneralp (2006) reports the analysis of Lotka-Volterra model with eigenvalue elasticities and concludes that the results are “surprisingly” in agreement with those of pathway participation metrics, although with some exceptions.

Why do the results of the two seemingly different approaches converge in some cases? Is this just a random phenomenon? Are there any classes of models for which pathway participation metrics and eigenvalue elasticity analysis make the same conclusions?

In the first part of this paper, we lay out some of the theoretical differences and similarities between aspects of pathway participation metrics (PPM) and eigenvalue elasticity analysis (EEA) to explain how the two approaches, despite their differences in the process of model analysis, can produce “similar and comparable” results under certain conditions. The second part of the paper presents applications of pathway participation metrics in four well-known models that have previously been analyzed using the eigenvalue elasticity approach. The case studies highlight some of the similarities and differences between the two approaches in detecting the dominant structure.

THEORETICAL COMPARISON OF EEA AND PPM

Telling systems-level stories in the pathway participation metrics approach begins with the variable of interest and it’s over time behavior. It detects the dominant feedback loops and pathways that are mainly responsible for the pattern of behavior observed by the variable of interest. Model analysis in eigenvalue elasticity approach starts with the eigenvalue of interest, a global measure of the system derived from model parameters and indicates the modes of the system behavior. The dominant structure is selected based on an elasticity metrics, eigenvalue elasticities, that determine the sensitivity of the behavior modes to the feedback structure.

The pathway participation metrics approach strives to arrive at global and system-level statements about the structure from local indicators such as the variables of interest and its overtime behavior revealed by simulation. The eigenvalue elasticity approach, on the other hand, attempts to find its way from global system-level measures -- originally developed for linear systems-- to explain simulation results often generated by nonlinear structure.

Despite the fact that the two approaches begin from different ends in the process of model analysis, the theoretical similarities between them can potentially lead to similar results in model analysis. One way to study the similarities and differences between pathway participation metrics and eigenvalue elasticity analysis is to compare and contrast the methods from the following three perspectives: 1) metrics used to characterize structure and behavior, 2) selection criteria for detecting dominant structure, and 3) interpretation of results.

1. Metrics

To identify dominant structure, we need to be able to characterize structure and behavior and establish a connection between the two. In PPM analysis, pathway participation metrics characterize the structure and total participation metrics characterize the behavior of the variable of interest. In EEA, eigenvalues characterize the behavior, and the connection between behavior and structure is established through elasticities (parameters, or gains). These measures meet in steady states. Appendix A provides a detailed technical discussion on the relationship between these metrics in the steady states.

1. 1. For linear systems, in the steady state condition, the total participation metrics, defined as sum of the participation metrics coming into a state variable, is equal to the dominant eigenvalue of the system.

$$\textbf{\textit{Total Pathway Participation Metrics}} = \textbf{\textit{Dominant Eigenvalue}}$$

1. 2. For linear systems, in the steady state condition, the participation metrics for the pathway that leaves a state variable and reaches the state variable of interest is equal to the elasticity of the dominant eigenvalue with respect to the parameter that connects the two state variables divided by the participation factor for the state variable of interest.

$$\textbf{\textit{Pathway Participation Metric}} = \frac{\textbf{\textit{Eigenvalue Elasticity}}}{\textbf{\textit{Participation Factor}}}$$

The participation factor of a state variable measures the degree of participation of the state variable in the (dominant) eigenvalue (for more details, see Eberlein, 1982).

For oscillatory systems, the detection of dominant structure in PPM, in its current implementation in Digest[1], is based on the rates of contractions and expansions in pathways, while in EEA the dominant structure is chosen according to the impact of the links and loops on the periodicity and envelop-curves of oscillatory modes. For explaining the periodicity and

[1] *Digest* is a piece of experimental software that detects and displays the dominant structure.

envelop-curves of observed cycles in oscillatory systems two new measures are developed which are determined by the cycles in pathway participation metrics: pathway frequency factors and pathway stability factors[2]. Pathway frequency factors indicate the participation of a pathway in periodicity of the observed cycles in the behavior of interest. Pathway stability factor indicates the participation of a pathway in the rate of divergence or convergence of the observed cycles.

1.3. In For linear systems, in the steady state condition, the total pathway stability factor, defined as sum of the stability factors for the pathways coming into a state variable, is equal to the real part of the dominant eigenvalue. Similarly, the total pathway frequency factor, defined as sum of the frequency factors for the pathways coming into a state variable, is equal to the imaginary part of the dominant eigenvalue.

Total Pathway Frequency Factors = Imaginary Part of Dominant Eigenvalue

Total Pathway Stability Factors = Real Part of Dominant Eigenvalue

Total pathway frequency factor reflects the length of an observed cycle and total pathway stability factor indicates how fast a cycle is moving towards or away from equilibrium.

It should be emphasized that the relationship between eigenvalues and pathway participation metrics holds only for the dominant eigenvalue of linear systems and in the steady states. These relationships may also hold for nonlinear systems around their equilibrium values if linearization does not significantly change the characteristics of the system.

2. Selection criteria for detecting dominant structure

In addition to metrics that characterize the behavior and the structure, the criteria for selecting dominant structure will also influence the outcome. Forrester (1982) considers a loop as dominant if it contains only links with large eigenvalue elasticities. Forrester identifies important links that connect state variables, based on their eigenvalue elasticity, to form the dominant feedback loops. "In most cases the dominant links fit together to form one or more interconnected feedback loops" (Forrester, 1982). The loops that are formed with important links are considered dominant "because they form paths for the propagation of waves" (Forrester, 1982).

The PPM approach takes a similar method that is identifying the dominant feedback loops based on the dominant pathways. The main difference is that the PPM analysis begins with the variable of interest, determines the dominant pathway and systematically follows back until a feedback loop is formed. In other words, in the first stage, the dominant pathway coming to the variable of interest is selected based on pathway participation metrics. In the second round, the states variable at the tail of the dominant pathway is considered the variable of interest and the process continues until a feedback loop is closed.[3]

[2] The pathway stability factor and pathway frequency factor are in fact the real and imaginary parts of PPM expressed in terms complex numbers.

[3] Other approaches for loop dominance identification based on the eigenvalue elasticities are developed that should be reviewed and compared with PPM approach for detecting dominant structures.

3. Interpretation of results

The concept of dominant structure is vast and open to different interpretations (Richardson, 1986). Researchers focusing on model analysis are under tremendous pressure to respond to a variety of needs and definitions of dominant structure. Some modelers who strive to formulate effective policies may expect model analysis to assist in detecting leverage points. A second group of modelers may be interested in explaining why the model does what it does in terms of its feedback structures. A third group of modelers may wish to identify a set of parameters that helps to fit the model output to historical data. Yet another group of modelers may be interested in model simplification and therefore look for a subset of the structure that does the same thing as the full structure.

While there can be a considerable overlap among these inquiries, the answer to one inquiry may not necessarily be the same as another. It is unlikely that a single method or technique can provide appropriate answers to all these questions. There is a need to clearly define the dominant structure in order to correctly set the expectations.

The question of what part of the structure gives rise to the behavior may be different than inquiries for leverage points—the part of the structure that changes the behavior. For example, balancing major loops usually generates oscillation; however, a minor balancing loop can dampen the oscillation. In other words, what creates the behavior may not be the same thing as that controls it.

Another point to keep in mind is that the dominant structures may or may not be the same as what is causing the particular pattern of behavior. It could merely mean that among the feedback loops that are responsible for creating the behavior of interest, the identified feedback loops are more influential. For example, in the Yeast model discussed in the next section, a major balancing loop around Cell deaths is identified as dominant during the reinforcing decline of the Cells. This major loop is, in fact, not the cause of the collapse although it is more influential than other feedback loops.

The main objective of the pathway participation metrics approach to model analysis is to tell correct and consistent system-level stories about the observed behavior of a simulation. In other words, it facilitates the interpretation of simulation output based on the stock and flow and feedback structure that created the behavior. The hope, however, is that the explanation of the dynamic behavior based on its feedback structure will help to search more effectively for answers to other inquiries in model analysis.

APPLICATION OF PPM IN FOUR WELL-KNOWN MODELS

1. The Inventory-Workforce Model

The Inventory-Workforce model is a simple second-order system that produces oscillatory behavior. In the model, Inventory is increased by production and reduced by sales. Production is proportional to workforce that is changes by net hiring which is determined by desired production, productivity, current workforce and a hiring adjustment time. The desired production is proportional to the inventory gap and expected demand which is the average demand or sale. The behavior of the model in response to a sudden change in sales is analyzed and explained using different approaches. While Mass & Senge (1975) strive to provide intuitive explanation for how the structure of the model creates oscillation, Gonçalves (2006) uses eigenvalue and eigenvector analysis to describe the cyclical behavior in the model. The former method explains the cycles based on rates of contractions and expansions in the time segments of oscillations. The latter approach identifies the feedback loops that contribute to the periodicity and envelope curve of cycles as a whole. In this section, we examine the behavior of the Workforce-Inventory model with PPM to provide both explanations for the oscillatory behavior in the model.

Phase Based Dominant Structure:

The PPM approach, in its current implementation in *Digest*, detects the dominant structure in oscillatory systems based on the four phases of the oscillation and the rate of expansions and contractions in pathways. Figure 1 depicts the feedback structure and the behavior of the Workforce-Inventory model as well as the dominant structure detected in PPM analysis.

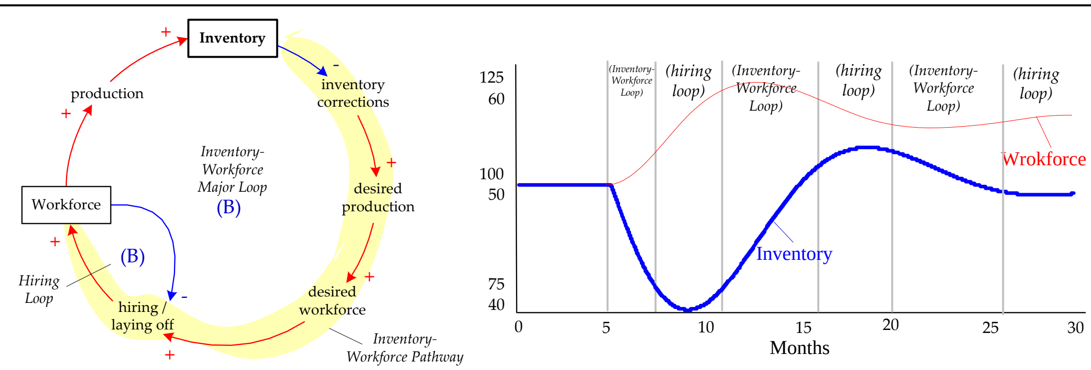

Figure 1: Dominant Structure in Phases of Inventory Dynamics based on PPM

The state variable Inventory contains one pathway that starts with Workforce and ends with Inventory. Workforce contains three pathways. The first is Inventory-Workforce Pathway that connects Inventory to Workforce through inventory corrections, desired production, desired workforce and hiring/laying off. The second pathway is actually a minor feedback loop -- Hiring Loop -- around Workforce. Finally, the third pathway connects the Expected Demand to the Workforce through desired production, desired workforce and hiring/laying off. The latter pathway is not shown in Figure 1 because Expected Demand is constant, except for a very short period of time around time 5 when demand is stepped up.

Time (months)	5	7.87	10.67	16.66	20.35	26.19	32.03
Duration (months)		2.87	2.8	5.99	3.69	5.84	3.66
Dominant Pathway for Inventory		Workforce - Inventory Pathway	Workforce - Inventory Pathway	Workforce - Inventory Pathway	Workforce - Inventory Pathway	Workforce - Inventory Pathway	Workforce - Inventory Pathway
Dominant Pathway for Workforce		Inventory-Workforce Pathway	Hiring Loop	Inventory-Workforce Pathway	Hiring Loop	Inventory-Workforce Pathway	Hiring Loop

Table 1: Dominant Pathways for Workforce According to PPM

Since the variable of interest, Inventory, contains only one pathway, the dominant structure will depend on Workforce. Table 1 shows the dominant pathways for Inventory over time determined by pathway participation. The first row in Table 1 is time (months), the second row shows the duration (month), the third and fourth indicates which pathway is dominant for Inventory and Workforce according to PPM.

The Story about Inventory Dynamics: According to PPM, after a sharp increase in demand at time 5, the Inventory-Workforce Balancing Major Loop has the highest influence for about 2.87 months. During this period, the increasing gap between desired and actual level of inventory along a higher expected demand leads the desired workforce to grow rapidly. Because of the growing desired workforce, the impact of Inventory-Workforce Pathway remains higher than the Hiring Loop. As a result, the Inventory-Workforce major loop is dominant. The slowing growth in the desired workforce diminishes the impact of Inventory-Workforce pathway on Workforce and the Hiring Loop dominates at time 7.87. After 2.8 months, when the desired workforce begins to experience a speedy decline due to decreasing inventory corrections, the impact of the Inventory-Workforce Pathway on Workforce exceeds that of the Hiring Loop, and thus the Inventory-Workforce Major Loop becomes dominant at time 10.66. The major loop remains dominant for about 6 months. Once the system reaches steady states, the major loop would be dominant for 5.84 months while the minor Hiring Loop would dominate for about 3.66 months, and the story repeats with the same periodicity.

Dominant Structure in Observed Cycles:
The two news measures for analyzing the characteristics of observed cycles, pathway stability factor and pathway frequency factor, can be calculated given the information about cycles in pathway participation metrics. Pathway frequency factors indicate the participation of a pathway in periodicity of the observed cycles in the behavior of interest. The larger the pathway frequency factor means the smaller the periodicity. On the other hand, pathway stability factor indicates the participation of a pathway in the rate of divergence or convergence of the observed cycles. A negative pathway stability factor means convergence while a positive pathway stability factor denotes divergence.

The new measures of stability and frequency indicates that the major loop is responsible for the cyclical behavior of Inventory and Workforce, however, the hiring minor loop around Workforce is the cause for convergence of the oscillations.

Half-Cycles	Factors	Workforce Inventory-Pathway	**Total**
Half-Cycle 1 (10 months)	Freq.	0.32	**0.32**
	Stab.	-0.12	**-0.12**
Half-Cycle 2 (9.5 months)	Freq.	0.33	**0.33**
	Stab.	-0.127	**-12.7**

Table 2: Inventory Frequency and Stability Factors

Half-Cycles	Factors	Inventory-Workforce Pathway	Hiring Loop	ExpDemand -Workforce pathway	**Total**
Half-Cycle 1 (8 months)	Freq.	0.2	-	0.2	**0.40**
	Stab.	0.01	-0.2	-	**-0.19**
Half-Cycle 2 (9.6 months)	Freq.	0.34	-	-0.01	**0.33**
	Stab.	0.13	-0.25	-	**-0.12**
Half-Cycle 3 (9.5 months)	Freq.	0.33	-	-	**0.33**
	Stab.	0.123	-0.25	-	**-0.127**

Table 3: Workforce Frequency and Stability Factors

Table 2 and 3 provides the information on pathway stability factors and pathway frequency factors for Inventory and Workforces. As shown in Table 2, two half-cycles are identified in Inventory from the simulation outputs. In both half-cycles, Workforce-Inventory pathway, the only pathway involved in the Inventory, is dominant. Table 3 shows that in the first half-cycle of Workforce, 8 months, both Inventory-Workforce and Expected Demand-Workforce pathway share the dominance in creating the periodicity of Workforce. Therefore, the major feedback loop around Workforce and Inventory along with the minor loop around Expected Demand contributes to the periodicity of the first cycle. In the second cycle, the Expected Demand-Workforce Pathway plays no roles in the periodicity of Workforce and thus, the major loop becomes the only source of oscillation. The stability of the cycles comes from the minor loop around Workforce.

Notice that the total frequency factor for Inventory (Table 2) and Workforce (Table 3) converges in the steady states and is equal to the real part of the dominant eigenvalue. Similarly, the total stability factor for Inventory (Table 2) and Workforce (Table 3) converges in the steady states and is equal to the imaginary part of the dominant eigenvalue.

2. The Yeast Model

The Yeast model has been examined by Saleh (2002), Güneralp (2005) and Phaff et al (2006) using eigenvalue analysis as well as Ford's behavioral approach. The model consists of two stocks labeled Cells and Alcohol and shows that the impact of alcohol concentration on cell deaths and cells births can create an overshoot in the number of Cells. Figure 2 depicts the structure of the model that includes the impact of alcohol concentration in cell deaths. The simulation results indicate that the Cells population at time 65.5 reaches its maximum and then begins to decline.

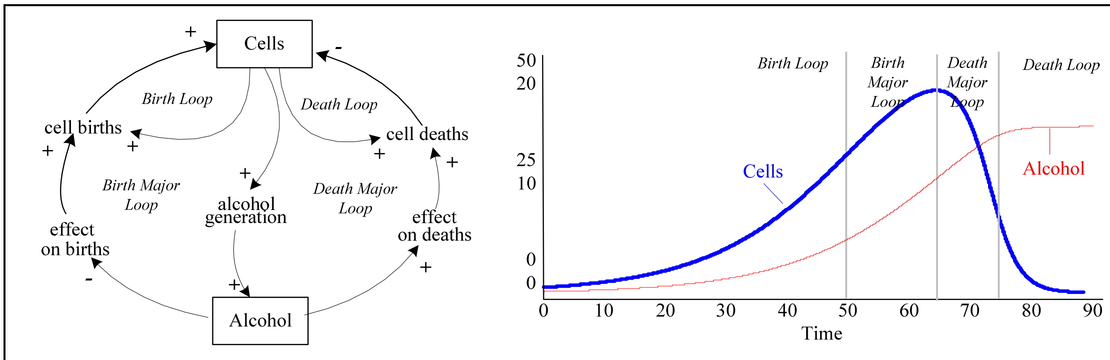

Figure 2: Dominant Feedback of the Yeast Model based on PPM

According to Phaff et al (2006), both Ford's behavioral approach and EEA indicate that the reinforcing cell Births Loop is responsible for the reinforcing growth in the beginning of simulation. Both methods also find the balancing decline towards the end of the simulation is generated by the balancing Death Loop. These findings are intuitively sound as both in the beginning and towards the end of the simulation the impact of Alcohol on births and deaths is almost constant or moves very slowly.

As shown in Figure 2, the application of PPM shows that the minor Birth Loop is more influential until time 51. From time 51 to time 65 the most influential loop, according to PPM, is the Birth

Major Loop. The dominant loop then shifts to the Death Major Loop between times 65 and 75, and after time 75 the minor Death Loop dominates.

It should be observed that PPM maintains the reinforcing minor Birth Loop as dominant as long as the reinforcing growth continues in the number of Cells. When the impact of balancing Birth Major Loop exceeds that of the reinforcing minor Birth Loop, the balancing Birth Major Loop dominates and the behavior of Cells alters to balancing growth. The Birth Major Loop remains dominant until the Alcohol level approaches a threshold and begins to influence cell deaths rate exponentially (as formulated in model). As a result, the balancing Major Death Loop dominates around time 65. The minor balancing Death Loop dominates around time 75 when the impact of Alcohol on death reaches a plateau.

The story told by PPM is generally consistent with EEA but departs in details and reasoning particularly during the transition periods of the behavior of Cells variable. In EEA, the overshoot in Cells is seen as oscillation in disguise. As a result, the Major Birth Loop is considered to be dominant around time 40 while the number of Cells is growing at an increasing rate. The dominant loops are then identified based of the contribution of the feedback loop to the oscillatory modes derived from the linearized systems (Saleh, 2002; Guneralp, 2006).

The shifts in loop dominance identified using Ford's behavioral approach, according to Phaff (2006), appears to be more consistent with the PPM story, particularly in details. According to Phaff (2006), Ford's approach identifies the Birth Loop as dominant until time 50. In phase 2, time 50 to 65, the Birth Major Loop is dominant, and in phase 3, time 65 to 75, both birth and death major loops are dominant.

It should be noted that the shifts in the dominant loops, according to pathway participation metrics, occurs at time 65 which coincides with the reinforcing decline in the behavior of Cells. It does not, however, mean that the balancing Death Major Loop is the "cause" for reinforcing decline. Both birth and death major loops contribute to the reinforcing decline, although the contribution of the Death Major Loops is more than the Birth Major loops.

In fact, the behavior of the model would not qualitatively change, if we eliminate the Death Major Loop. If Alcohol were to only impact the births, the number of cells would still follow the overshoot and collapse pattern of behavior although with slower dynamics. The story told by the pathway participation metrics will remain the same except in phase three when Cells decline with an increasing rate. In this case, the reinforcing decline in Cells, according to PPM, is induced by the same feedback loop that limits cell births. The increasing level of Alcohol continues to control cell births and when the births fall bellow deaths, Cells begin to experience a reinforcing decline.

3. The Case of the Industrial Structure Model

Another simple model that produces overshoot behavior is the Industrial Structure model that has been analyzed through PMM (Mojtahedzadeh, et al, 2004) and EEA approach (Kampmann et al, 2006) to identify the shifts in loop dominance. The model consists of two stocks -- Industrial Structures and Water Reserves. The continuous growth of Industrial Structures reduces Water Reserves and that diminishes Water Adequacy which, in turn, limits expansion of Industrial Structures.

Kampmann et all (2006) compare the results of EEA and PPM analysis of the Industrial Structure model. They found that dominant structure identified by the two approaches are "to a large extent, in accordance" with one another. The only point of departure between the two analyses is the duration in which Industrial Structures experiences a reinforcing decline (Figure 3).

> "Indeed, we question whether it makes sense to speak of two distinct phases in this case—the time path of the variables is simply a temporary transition not driven by structure but by the relative position of the state variables" (Kampmann et al, 2006).

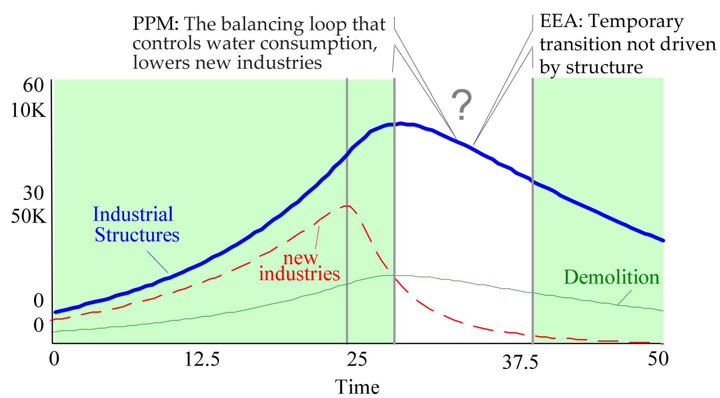

Figure 3: Explaining Reinforcing Decline in Overshoot

The pathway participation metrics suggests the same balancing structure around Water Reserves that controls the New Industrial Structure causes the reinforcing decline.

> "what forces Industrial Structure to fall faster and faster is exactly the same process that controls it. The balancing loop that controls water consumption continuously lowers new industries and, once new industries fall behind industrial demolition, Industrial Structure generates a reinforcing decline" (Mojtahedzadeh et al, 2004).

The different interpretation by two approaches is not unexpected in the transition period. The idea of *"temporary transition not driven by structure"*, however, raises the question whether it is possible to explain the temporary transition of complex systems in terms of their feedback structure. Is there a class of models whose temporary transition can be explained by its feedback structure?

Despite the similarity in the Industrial Structure and Yeast models, EEA provides two very different explanations underlying the feedback structure influential in creating the overshoot and collapse. In the yeast model, the transition period in the overshoot is seen as oscillation while in the case of the Industrial Structure model the reinforcing decline phase of overshoot is explained by temporary transition caused by relative positioning of stocks and not driven by feedback structure.

If temporary transitions cannot be explained with the feedback loop structure, what should be the metrics for identifying them and analyzing them? Is the "relative position" of the state variables sufficient for the analysis of the transition periods? What other "building blocks" are needed, in addition to feedback loops, to fully explain the patterns of behavior from the beginning of the simulation to the end.

The PPM approach suggests the concept of "pathways" in addition to feedback loop, to characterize the transition period. Pathway participation metrics take into account the "relative position" of the state variables in the form of the ratio of the net changes in the two state variables. The ratio of the net changes becomes the elements of eigenvectors in the steady states when the dynamics of the *relative position* of state variable settles and the moves toward the dominant eigenvalue (Mojtahedzadeh et al, 2004). Pathways not only carry the information about the "relative position" of state variables but also the polarity of the links through which this information makes its way around the feedback loop containing the variable of interest. In the extreme case when the polarity of a pathway is zero, the "relative position" of state variables does not impact the variable of interest.

4. The Prey-Predator Model with Prey Starvation:

The Prey-Predator model is a second-order nonlinear system that produces sustained oscillation. The addition of the prey starvation to the model bounds the growth of prey and dampens the oscillation. In this model, as shown in Figure 4, prey birth and natural death is proportional to prey population size and prey starvation grows faster than the prey population. The number of preys killed by predators and predator births depends on the size of the both population.

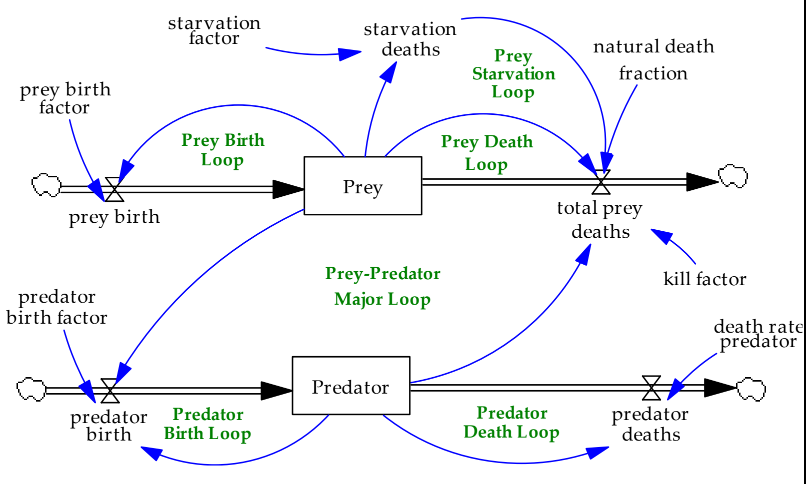

Figure 4: Prey-Predator Model with Prey Starvation

Phase Based Dominant Structure:

Figure 5 shows the behavior of Prey and Predator and its phases based on shifts in the dominant structure. The behavior of Predator contains four phases. In the first and third phase, the Predator birth loop and death loop is dominant, respectively. In phase 2 and 4 the Prey-Predator pathway is more influential than others in creating the behavior of Predator. The behavior of Prey in the beginning and end part of phase 2 and 4 is driven by Predator-Prey pathway, and therefore the major loop around Prey and Predator is dominant. In the middle part of phases 2 and 4, the impact of the Prey Starvation Loop in creating the dynamics of Prey exceeds other loops and pathways.

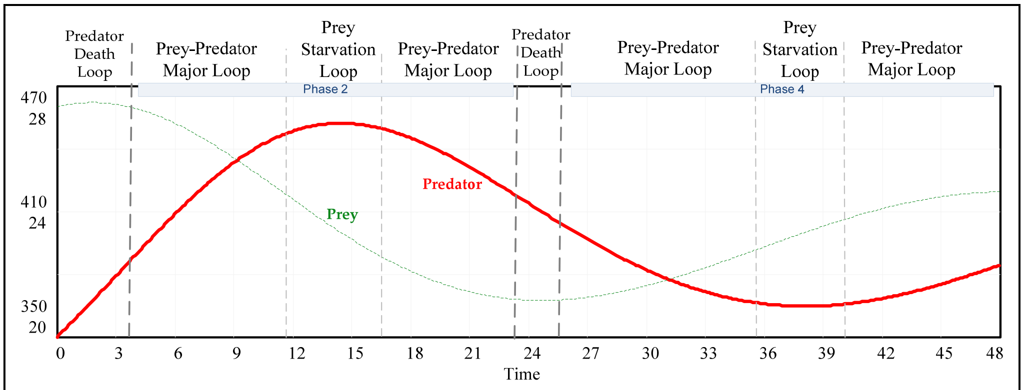

Figure 5: The Behavior of Predator and its Phases

Description of the Phase in Predator:

Phase 1(from 0 to -3.5): The rapid growth in Predator during this phase is mainly influenced by the reinforcing Predator Birth loop as Prey is relatively constant.

Phase 2 (from 3.5 to 24): As Predator grows, more preys are killed, therefore Prey begins to decline quickly and the Predator-Prey Pathway causes further decline in Prey. At the same time, the falling Prey controls the growth of Predator causing Predator to grow slower and slower. Around time 11.5, when Predator is changing very slowly, the dynamics of Prey is highly influenced by the balancing Prey Starvation Loop. Around time 17, Predator begins to fall quickly due to declining Prey and the major balancing loop dominates again and controls the rate of decline in Prey.

Phase 3 (from 24 to 25.5): In this phase, while Prey remains almost constant, the Predator balancing death loop control the decline of Predator.

Phase 4 (from 25.5 to End): This phase is the reverse of what happened in Phase 2. The major balancing loop is dominant for 11.5 months controlling the decline of predator and speeding the growth of Prey. Around time 37 when the changes in Predator is slow Prey Starvation Loop slows down the growth of Prey. However, around time 40 the major balancing loop, Prey-Predator Loop dominates causing a reinforcing growth in Predator and a balancing growth in Prey.

Dominant Structure in Observed Cycles:

Table 4 and 5 summarizes the pathway stability factor and pathway frequency factor for both Prey and Predator.

Half-Cycles	Factors	Prey-Predator Pathway	Predator Minor Loops			**Total**
			Total	*Birth*	*Death*	
Half-Cycle 1	Freq.	0.133	-	-	-	**0.133**
(23.7 months)	Stab.	-0.028	-0.017	*0.183*	*-0.2*	**-0.045**
Half-Cycle 2	Freq.	0.134	-	-	-	**0.134**
(23.5 months)	Stab.	-0.022	0.01	*0.19*	*-0.2*	**-0.012**
Half-Cycle 3	Freq.	0.133	-	-	-	**0.133**
(23.6 months)	Stab.	-0.025	-0.006	*0.194*	*-0.2*	**-0.031**
Half-Cycle 4	Freq.	0.134	-	-	-	**0.134**
(23.5 months)	Stab.	-0.023	0.003	*0.197*	*-0.2*	**-0.020**

Table 4: Predator Frequency and Stability Factors

Half-Cycles	Factors	Prey Minor Loops				Predator -Prey Pathway	**Total**
		Total	*Birth*	*Death*	*Starvation*		
Half-Cycle 1 (23.7 months)	Freq.	-	-	-	-	0.137	**0.137**
	Stab.	-0.059	*0.15*	*-0.01*	*-0.199*	0.022	**-0.037**
Half-Cycle 2 (24 months)	Freq.	-	-	-	-	0.131	**0.131**
	Stab.	-0.041	*0.15*	*-0.01*	*-0.181*	0.025	**-0.016**
Half-Cycle 3 (23.3 months)	Freq.	-	-	-	-	0.135	**0.135**
	Stab.	-0.052	*0.15*	*-0.01*	*-0.192*	0.023	**-0.029**
Half-Cycle 4 (23.7 months)	Freq.	-	-	-	-	0.133	**0.133**
	Stab.	-0.046	*0.15*	*-0.01*	*-0.186*	0.024	**-0.022**

Table 5: Prey Frequency and Stability Factors

Half-Cycles: According to Table 4 and 5, both Prey and Predator go through half-cycles that are approximately 23.5 months long. The total frequency factors for Prey and Predator is about 0.13 which reflects the length of half-cycles involved in these two variables. The total stability factors for both Prey and Predator are negative in all cycles indicating dampening oscillation in the behavior of the model. Note that total frequency and stability factors for Prey and Predator converge in the long run and are equal to the imaginary and real part of the dominant eigenvalue, respectively.

Periodicity (Frequency): The periodicity of Prey and Predator is driven by the major balancing loop around the two variables. According to Table 2 and 3, in the half-cycles of Predator, the frequency factor for the Predator-Prey Pathway in dominant, while in the half-cycles of Prey the frequency factor for the Prey-Predator Pathway is dominant. As a result, the Predator-Prey balancing Loop becomes the only source of oscillation.

Envelope Curve (Stability): The stability of the system is mainly driven by the balancing Prey Starvation Loop. The stability factors for Predator indicates that Prey-Predator pathway is dominant and the stability factors for Predator shows the significance of Prey starvation loop.

CONCLUSION

During the last decade, a number of techniques have been developed to detect dominant structure in dynamic models. Several comparative studies conducted to contrast the outcome of alternative techniques for model analysis have concluded that a significant overlap exists among the outcomes of various approaches to model analysis. This paper explored some of the similarities and differences between pathway participation metrics and eigenvalue elasticity analysis to explain the convergence and divergence of outcomes of the two methods in identifying dominant structure in dynamic models. It showed that the metrics used by PPM and EEA to characterize the structure and behavior are closely related in linear systems in steady states.

In analyzing the nonlinear systems, the eigenvalue approach relies on the piecewise linearization of the nonlinear system. The challenge is to assure that the linearized system is sufficiently accurate representation of the original nonlinear system under different conditions. The second case study shows that linearization of nonlinear systems that produce overshoot pattern of behavior can result in complex eigenvalue. The interpretation of complex eigenvalues in a system that does not oscillate is difficult.

Understanding the transient behavior presents another challenge in eigenvalue analysis approach as discussed in the third case study. Does the sub-dominant eigenvalues appropriately characterize the transition period? Does the analysis of sub-dominant eigenvalues correspond to simulation outcomes? Should we consider transient behavior as something that may not be driven by feedback structure? The pathway participation approach does not work with eigenvalues, although the metrics used in this method come close to the dominant eigenvalues and their elasticities in linear system in steady states. The total participation metrics that characterize the behavior corresponds to the simulation outputs both in transition periods and steady states.

Identifying dominant structure in oscillatory systems using PPM approach has so far been based upon time slices of the behavior of the variable of interest. The dominant structure is identified according to rates of contractions and expansions in variables. The method is similar to the intuitive explanation of oscillation provided by Mass and Senge (1975). An alternative approach in characterizing oscillations is based on frequencies and the envelope curves of observed cycles in simulation output. The two new metrics that are developed for oscillatory systems help to examine the characteristics of observed cycle in the simulation outputs. Pathway frequency factors indicate the participation of a pathway in periodicity of the observed cycles in the behavior of interest. On the other hand, pathway stability factor indicates the participation of a pathway in the rate of divergence or convergence of the observed cycles. While not thoroughly tested, it is expected that the dominant structure detected for the observed cycles using these new measures converge with that EEA in that steady states.

One challenge for both PPM and EEA approaches is to precisely and clearly define the concept of dominant structure and the set realistic expectations for what each method can offer. Is the dominant structure the smallest subset of the structure the produces the same behavior? Or, the partial structure around the system's parameters that results in significant change in the behavior is the dominant structure? Alternatively, one can define the dominant structure as a subset of feedback loops and pathways that helps to explain the behavior of the dynamic systems. The pathway participation approach to model analysis strives to construct correct and consistent system-level stories about the observed behavior of a simulation and to interpret simulation outputs based on the stock and flow and feedback structure that created the behavior.

Appendix A:

This Appendix describes the relationships between pathway participation metrics and eigenvalue elasticities. These relationships hold only for the dominant eigenvalue of linear systems and in the steady states. These relationships may also hold for nonlinear systems around their equilibrium values if linearization does not significantly change the characteristics of the system.

Two new measures for analyzing characteristics of observed cycles are formulated based on the reinforcing and balancing periods in pathway participation metrics: The pathway frequency factor that indicates the participation of a pathway in the periodicity of the observed cycles the pathway stability factor which indicates the participation of a pathway in the rate of divergence or convergence of cycles.

1. For linear systems, in steady-states, the pathway participation metrics for the pathway that connect j-th state variable to i-th state variable is:

$$\rho_{ij} = a_{ij} * r_j / r_i$$

where r_i and r_j and i-th and j-th right eigenvector associated with dominant eigenvalue and a_{ij} is an element of reduced form matrix.

Proof:
Consider the following linear system:

[1] $$\dot{X} = AX$$

Where X, (n*1), is the vector of state variable, $\dot{X}$, (n*1) is the vector of changes in the state variable, and A (n*n) is the interaction matrix. Write the dimension,.

Given the following dynamic equation for the i-th state variable:

[3] $$\dot{x}_i = \sum_j a_{ij} x_j$$

The PPM for the link that leaves state variable j and reaches state variable i, ρ_{ij} , will be:

[4] $$\rho_{ij} = a_{ij}\, \dot{x}_j / \dot{x}_i$$

In steady state, since by definition we have $\dot{x}_j = \lambda r_j$ and $\dot{x}_i = \lambda r_i$, thus ρ_{ij} approaches

[5] $$\rho_{ij} = a_{ij} r_j / r_i$$

where r_j and r_i are the j-th and i-th eigenvector associated with the dominant eigenvalue, λ, respectively.

1.1. For non-oscillatory systems, the value of ρ_{ij} approaches $a_{ij} r_j / r_i$ in steady states.
1.2. For oscillatory systems, ρ_{ij} can be expressed by complex numbers whose real part is s_{ij} and imaginary part is f_{ij}. Within a half-cycle, the duration in which the sign of ρ_{ij} remains the same is:

[6] $$\omega_{ij}^{R} = a\tan(-f_{ij}/s_{ij}) * \omega_i / \pi ,$$

where ω_i is a half-cycle of the oscillation which in steady states is $\pi / i\lambda$ and $i\lambda$ is the imaginary part of the dominant eigenvalue.

2. The sum of participation metrics coming to state variable i, ρ_{ij}, called total pathway participation metrics for state variable x_i, $T\rho_i$, equals the dominant eigenvalue.

Proof:

[7] $$\begin{aligned} T\rho_i &= \sum_j \rho_{ij} \\ &= \sum_j a_{ij} * r_j / r_i \\ &= \frac{1}{r_i} \sum_j a_{ij} r_j \end{aligned}$$

The definition of right eigenvalue is $\sum_j a_{ij} r_j = \lambda * r_i$, where λ is the (dominant) eigenvalue .
Therefore, the above equation can be rewritten as:

[8] $$\begin{aligned} T\rho_i &= \frac{1}{r_i}(\lambda * r_i) \\ &= \lambda \end{aligned}$$

2.1. For non-oscillatory systems, the sum of PPM, $T\rho_i$, is essentially the dominant eigenvalue in steady states.

2.2. For oscillatory systems, the total pathway participation metrics can be expressed in complex numbers whose real part and imaginary part are Ts_i and Tf_i, respectively. Within a half-cycle, the duration in which the sign of $T\rho_i$ remains the same is $\omega_i^R = a\tan(-Tf_i/Ts_i) * \omega_i / \pi$, which is the same as $a\tan(-i\lambda/r\lambda)/i\lambda$, where $r\lambda$ and $i\lambda$ are the real part and imaginary part of the dominant eigenvalue. If the duration is which $T\rho_i$ is greater (less) than half of the half-cycle, the variable is of interest show exploding (dampening) oscillation.

3. The participation metric for the pathway that connect the j-th state variable to i-th state variable, ρ_{ij} , is:

[9] $\rho_{ij} = (1/\lambda) * \varepsilon_{ij} / \kappa_i$

Where λ is the dominant eigenvalue, ε_{ij} is the elasticity of dominant eigenvalue, a_{ij} is the parameter that connects the j-th to the i-th state variable, and κ_i is the participation factor.

The Participation Factor, κ_i , for the i-th state variable shows the level of the participation of the i-th state variable in the (dominant) eigenvalue. (See Eberlein, 1982)

Proof:

We already showed that

[5]' $\rho_{ij} = a_{ij} r_j / r_i$

Multiplying both denominator and numerator of the right side of the equation by f_i and λ gives:

[10] $\rho_{ij} = a_{ij} r_j f_i \lambda / r_i f_i \lambda$

where f_i is the i-th element of the left eigenvalue associated with the dominant eigenvalue. The definition of the elasticity of the (dominant) eigenvalue with respect to a_{ij} is

[11] $\varepsilon_{ij} = f_i r_j a_{ij} / \lambda$

We also know that, y definition, the participation factor for the i-th state variable is:

[12] $\kappa_i = r_i f_i$

Substituting [11] and [12] in [10] yields:

[13] $\rho_{ij} = \lambda * \varepsilon_{ij} / \kappa_i$

4. For the i-th state variable, we have

[14] $\rho_{ij} / \rho_{ii} = \varepsilon_{ij} / \varepsilon_{ii}$, if ρ_{ii} and ε_{ii} are nonzero.

Poof:

Substituting [12] in the left side of [13] yields:

[15] $\left(\lambda * \varepsilon_{ij} / \kappa_i\right) / \left(\lambda * \varepsilon_{ij} / \kappa_i\right) = \varepsilon_{ij} / \varepsilon_{ii}$

5. The sum of all pathway frequency factors for all pathway coming to the variable of interest, is equal to the imaginary part of the dominant eigenvalue. Similarly, the sum of all pathway stability factors for all pathways coming to the variable of interest is equal to the real part of the dominant eigenvalue.

Proof:
Suppose ϖ_i is a half-cycle of the variable of interest, x_i, that starts and ends with $\dot{x}_i = 0$, the pathway frequency factor for the pathway that connects j-th to i-th state variable is defined by:

[16] $$f_{ij} = (\pi/\omega_i) * \tilde{\rho}_{ij} \big/ T\tilde{\rho}_i$$

where $T\tilde{\rho}_i$ is total participation metric and $\tilde{\rho}_{ij}$ participation metrics for the pathway that connects j-th to j-th state variable in the beginning of the cycle. The total frequency factor is:

[17] $$\begin{aligned} Tf_i &= \sum f_{ij} \\ &= \sum (\pi/\omega_i) * \tilde{\rho}_{ij} \big/ T\tilde{\rho}_i \\ &= (\pi/\omega_i) \end{aligned}$$

which in steady state is the imaginary part of the dominant eigenvalue.

On the other hand, the stability factor for the pathway that connects j-th to i-th state variable is:

[18] $$s_{ij} = f_{ij} \big/ \tan\left(-\pi * \omega_{ij}^R \big/ \omega_i\right)$$

The above equation can be rewritten as:

[19] $$\omega_{ij}^R = a\tan\left(- f_{ij} / s_{ij}\right) * \omega_i \ / \ \pi$$

A comparison between equations [19] and [6] indicates that the stability factor is the same as the real part of the pathway participation metrics when it is expressed in complex numbers. Therefore, the total pathway stability factor is equal to the real part of the total participation metrics. Consequently, the total pathway stability factor in steady states approaches the real part of the dominant eigenvalue.

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