Bifurcation Sequence in a Simple Model of Migratory Dynamics

by

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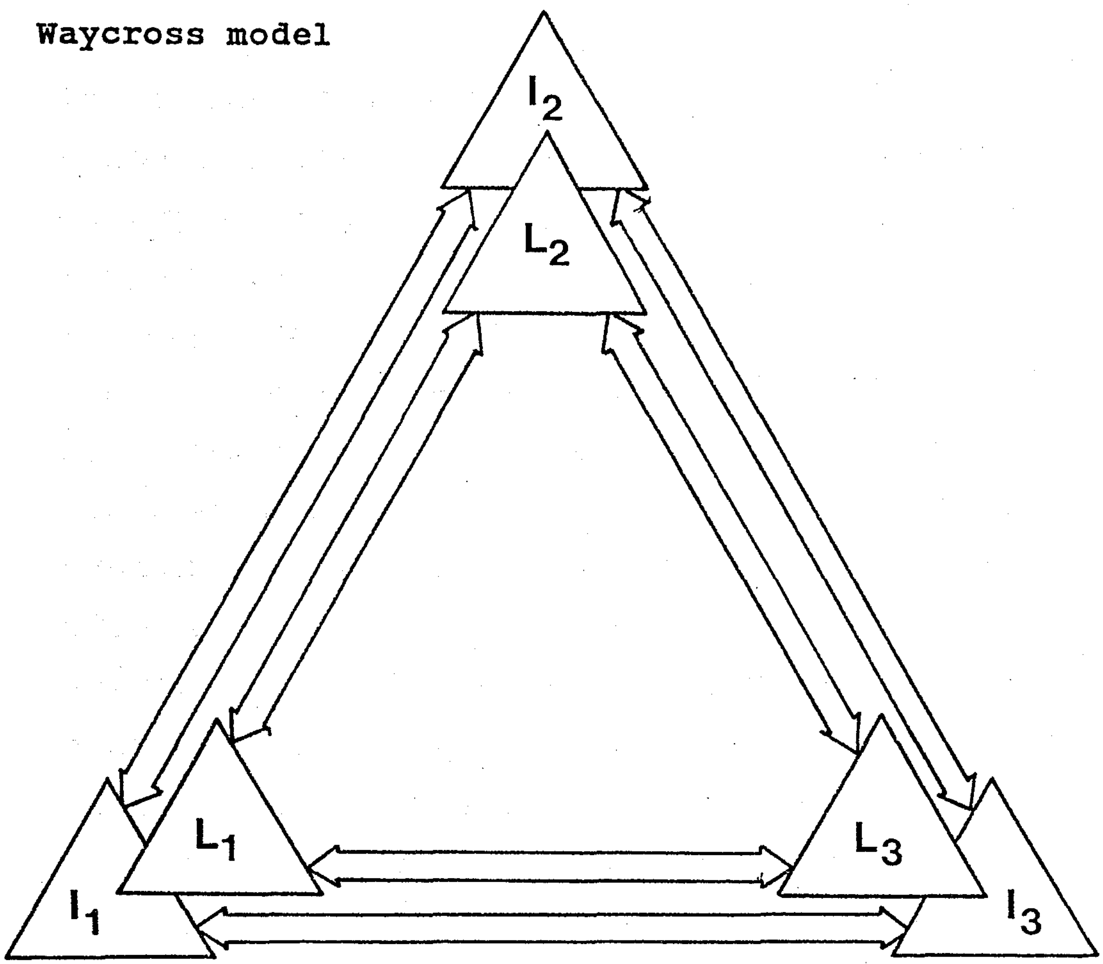
Abstract

A bifurcation sequence in the Waycross model is studied by means of Poincaré section techniques. The bifurcation parameter B is gradually reduced from 2.00 to 1.50. This parameter measures the inclination of one type of minority families (Lomanians) to move into districts with many families of another type of minority population (Itrachians). Because of symmetry the attractors in this 4-dimensional migratory model occur in pairs with opposite directions of cyclic population movements. A pair of simple limit cycle attractors are found to remain stable under formation of a pair of period-2 attractors. In a certain parameter range, the model thus contains four entangled attractors. We follow how the period-2 attractors become chaotic through formation and subsequent destabilization of 2-dimensional tori. On the way, regular period-14, period-18 and period-4 attractors are produced through frequency-locking. We thereafter observe a case of type III intermittency when the two period-1 orbits become unstable, and finally the two chaotic attractors merge with each other.

Waycross model

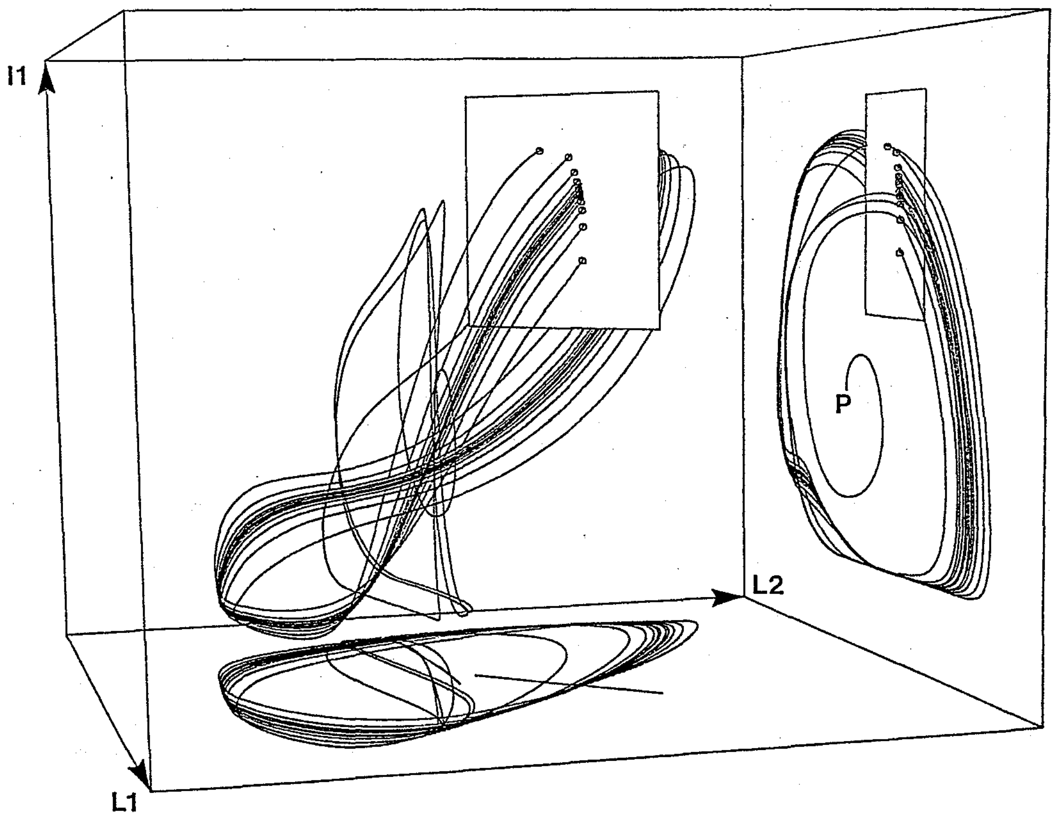


Dynamical Hypothesis

Unstable behavior in migratory systems arises from positive feed-back mechanisms which cause people to cluster in neighborhoods that already house relatively many families of similar characteristics or origin. The Waycross model considers two such subpopulations: Lomanians and Itrachians which can migrate between three districts of a town. A rotating behavior is introduced by assuming that while Lomanians are attracted to areas populated by Itrachians, the Itrachians do not particularly appreciate Lomanians. Once Lomanian families start to move into Itrachian areas, the Itrachians prefer to move out. The non-linear restraints required to keep the system within a finite volume of phase space derive from the logical condition that all populations must remain non-negative. Finally, the conservation of families in the migratory process ensures a relatively low dissipation. This facilitates the development of complicated non-linear phenomena such as deterministic chaos.

Since both the total Lomanian and the total Itrachian populations are assumed to be constant, the model only contains four independent variables: the number of Lomanian families in district 1 (L_1) and district 2 (L_2) and the number of Itrachian families in the same two districts (I_1 and I_2, respectively).

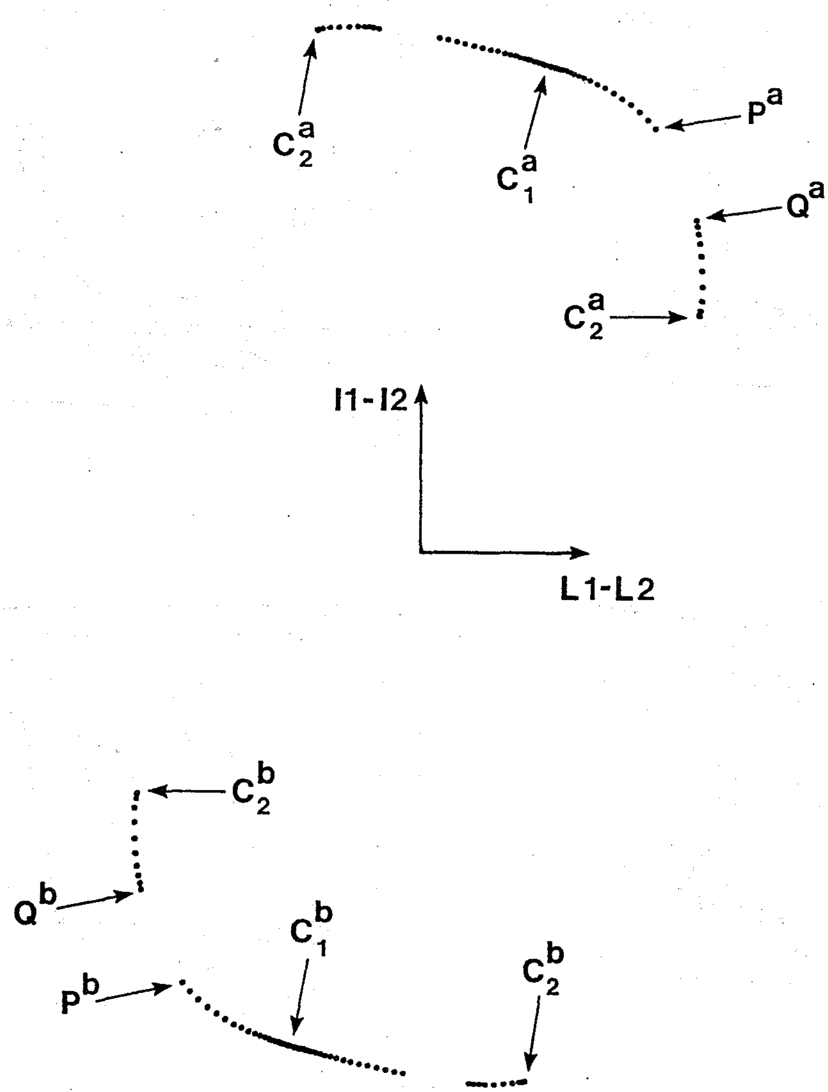
Because of symmetry, the attractors occur in pairs with opposite directions of cyclic population movements. Our investigation has revealed that a pair of simple limit cycle attractors which exist for B = 2.0 remain stable under the formation of a pair of stable period-2 attractors.



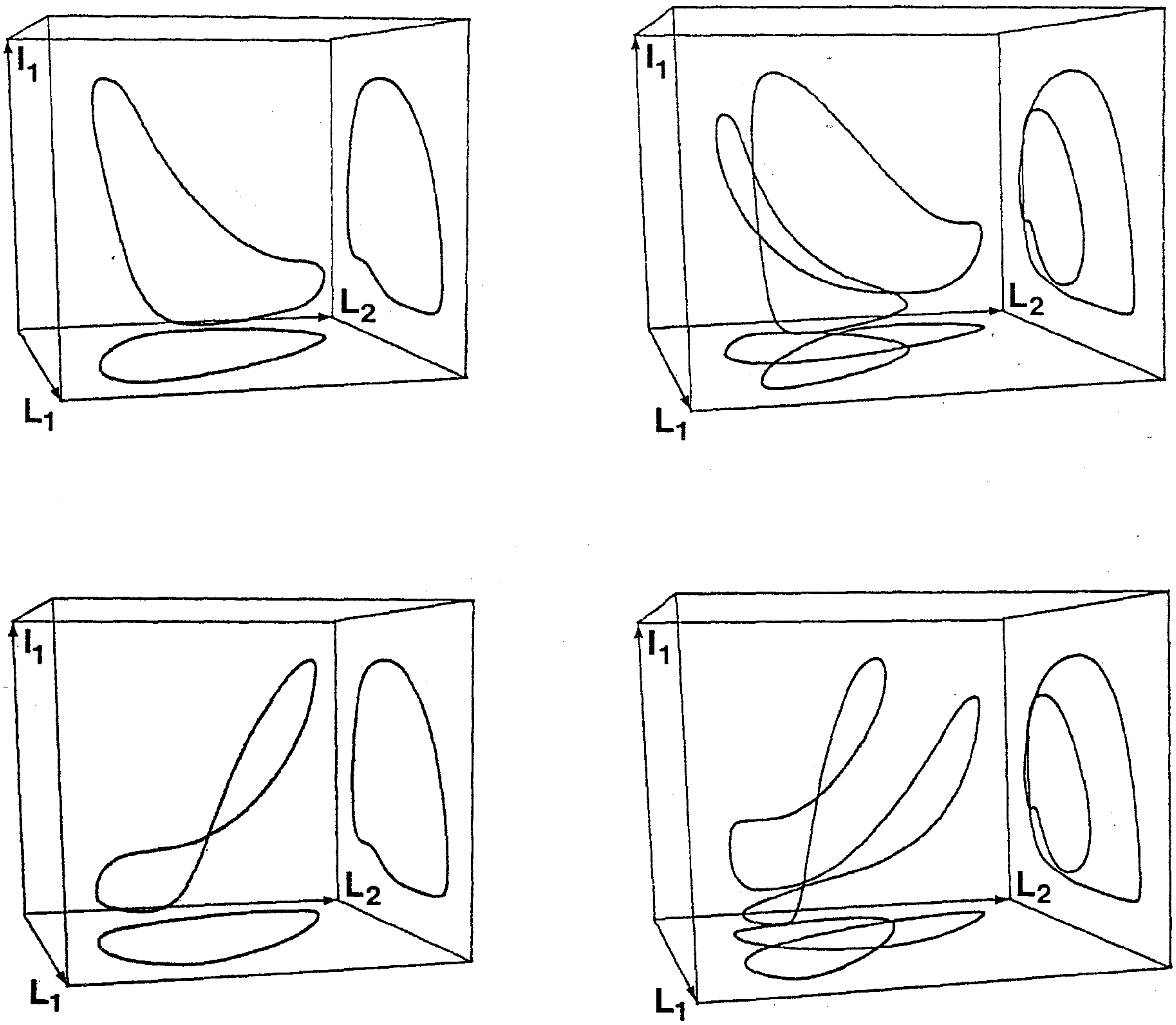
Poincaré section - B = 2.0

To investigate the model, we use Poincaré sections: The intersections between the trajectories in the four-dimensional phase space and a three-dimensional hyperplane are recorded for the trajectories passing through the plane in a particular direction. In this way, the system is reduced by one dimension. The figure illustrates the construction of a Poincaré section using a three-dimensional phase space and a regular two-dimensional plane.

Four periodic attractors - B = 1.8

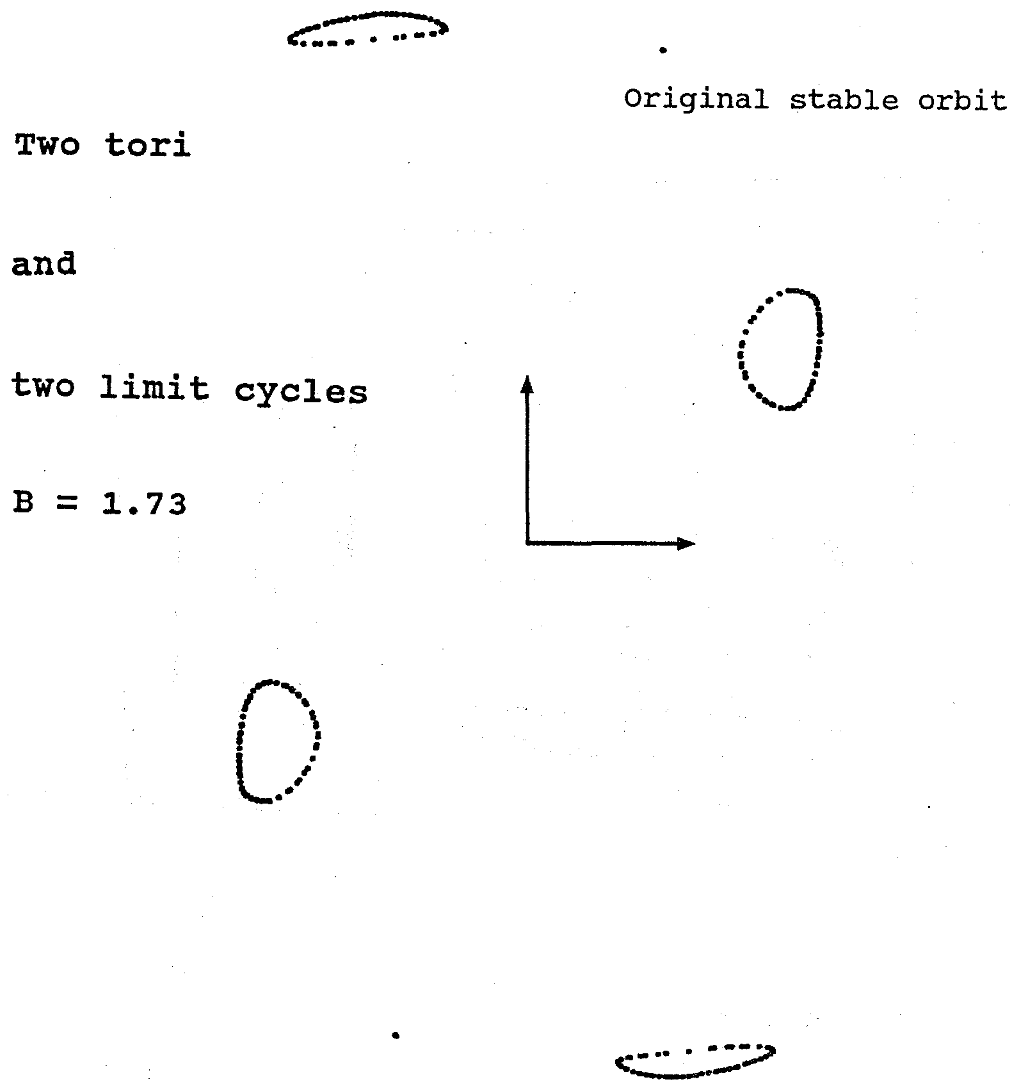


A two-dimensional projection of the Poincaré sections for B = 1.8 reveals four co-existing attractors. The three-dimensional cutting-plane passes through the unstable equilibrium point ($L_1 = L_2 = L_3 = I_1 = I_2 = I_3$) with a normal vector of (1,1,1,1). Transient approaches to the four attractors are seen beginning at P_a, P_b, Q_a and Q_b, respectively. The corresponding stable solutions are denoted C_1^a, C_1^b, C_2^a and C_2^b.

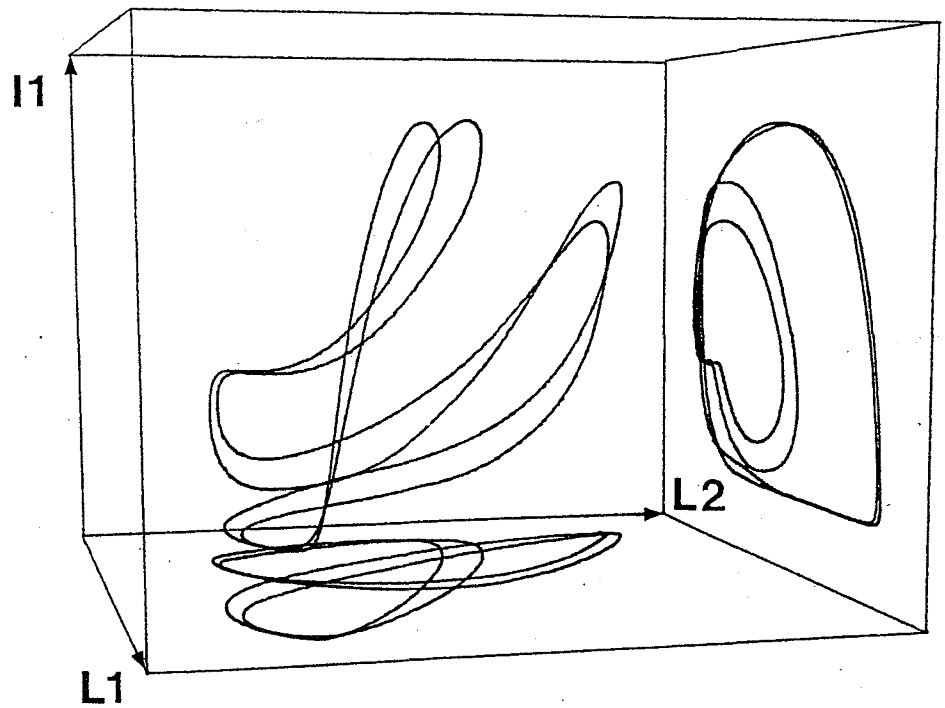


Four periodic attractors - B = 1.8

Three-dimensional representations of the four co-existing attractors for B = 1.8.

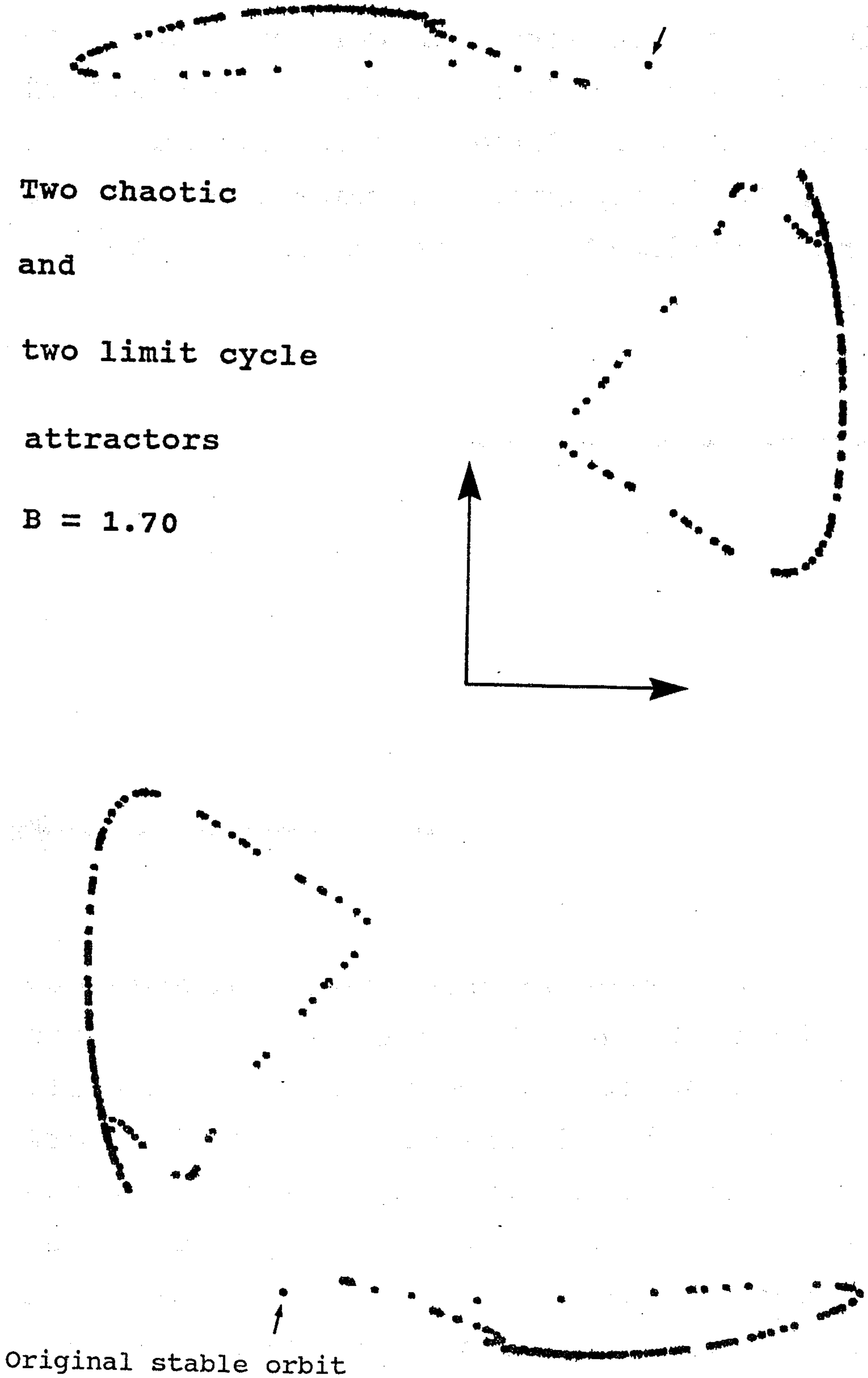


A two-dimensional projection of the Poincaré section for B = 1.73. The two symmetric period-1 orbits remain stable, while the two symmetric period-2 limit cycles each undergo a torus-bifurcation when B is reduced below 1.75. This means that the trajectories are located on the surface of a torus (or a doughnut). In this case the oscillations never repeat themselves. The solution is said to be quasi-periodic.



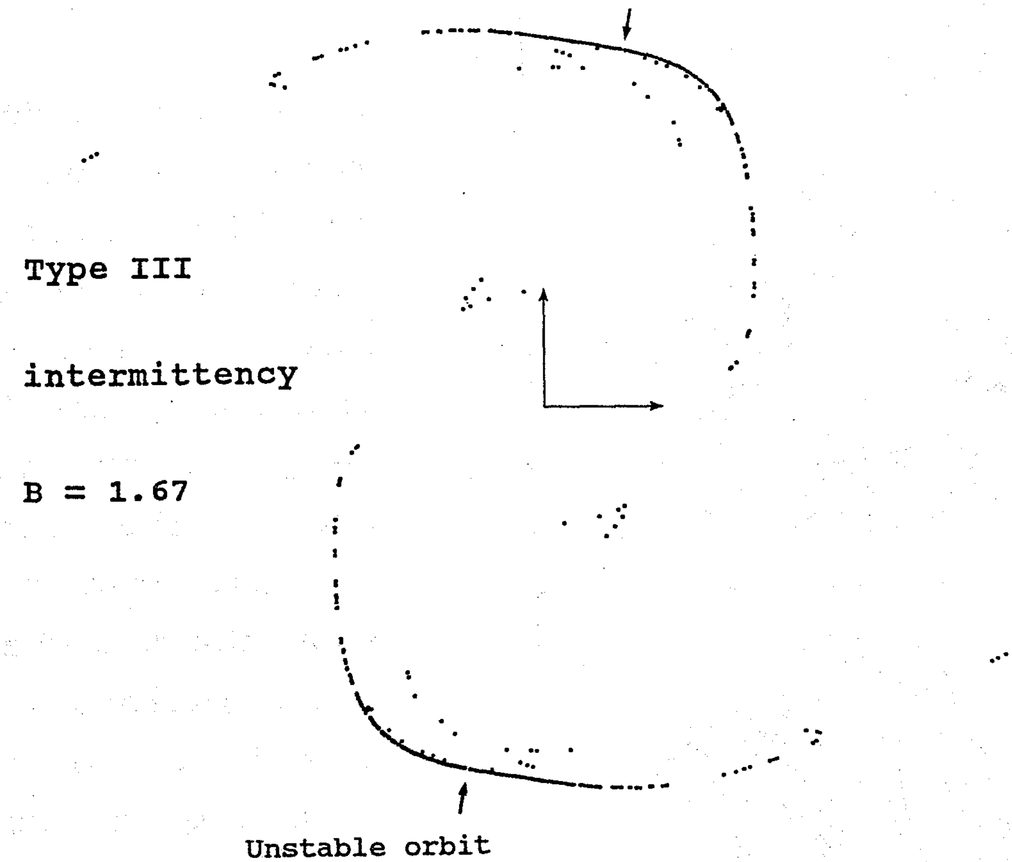
Frequency-locking - B = 1.72

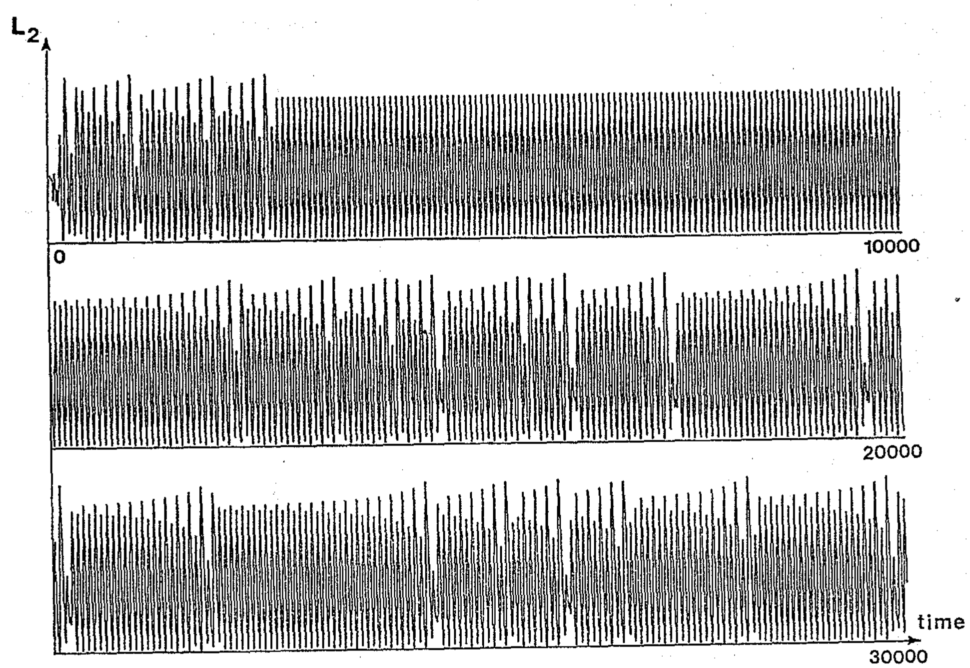
Sometimes the trajectories on the surface of the doughnut describe a closed curve. This is the case when B = 1.72, and it is a result of frequency-locking between the two frequencies associated with the two unstable modes that together produce the torus. For this parameter value the solution is of period 4. Other parameter combinations give different solutions, for example period 14 and 18. We are here observing part of a so-called devil's staircase.


Two chaotic
and
two limit cycle
attractors
B = 1.70
Original stable orbit

When B is reduced to 1.7, the tori become unstable, and the Poincaré sections exhibit folding. The trajectories can no longer be described as being located on the surface of a doughnut. Now these two attractors are chaotic. It is important to note that the two original period-1 orbits continue to be stable.

For B = 1.67, another qualitative change has taken place. By now the two period-1 orbits have turned unstable. As a result, type III intermittency is observed. In the Poincaré section, this can be seen as a merging of the two halves of each attractor, the merging being at the location of the unstable periodic orbits. In the time-plot, the intermittency is revealed as an alternating increase and decrease in amplitude of the oscillations.


Type III
intermittency
B = 1.67
Unstable orbit


L_2
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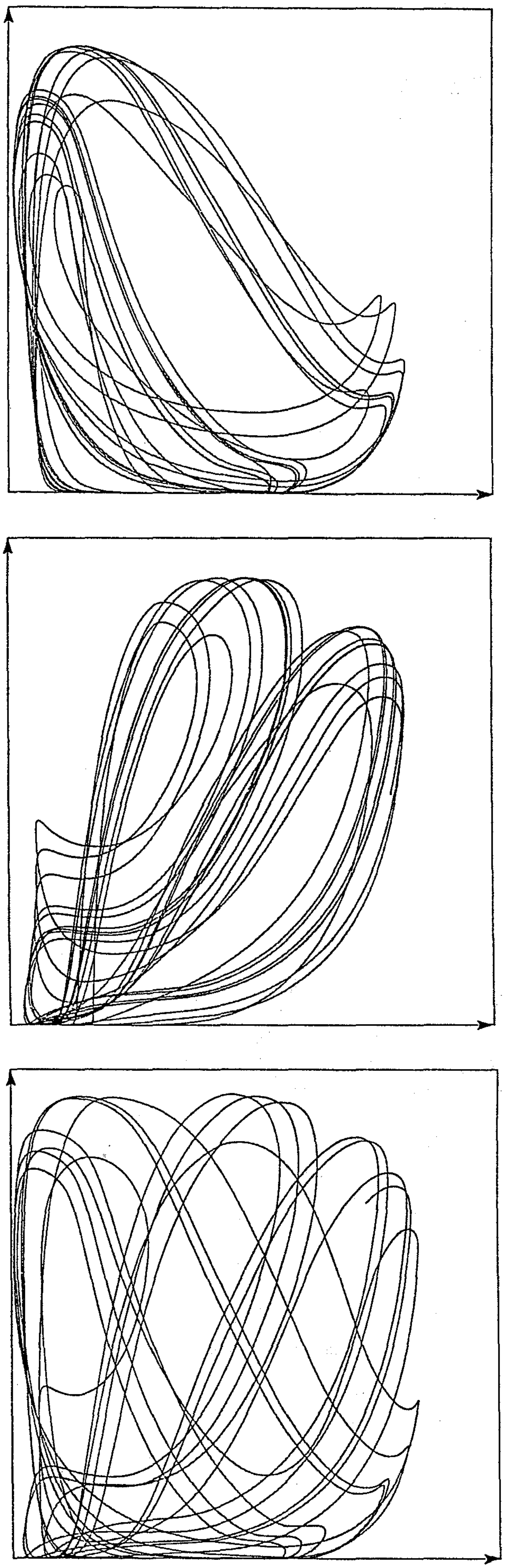

Merging of attractors

With even further reduction of B, the two chaotic attractors continue to expand, until they finally merge with each other. The first two pictures here show the two chaotic solutions when B = 1.7. In the last figure, qualitative features of the first two pictures can be seen, as a consequence of the merging. In this case B = 1.5.