Feedback loop gains and system behavior

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Introduction

The method of system dynamics has relied extensively on feedback loops to explain how system structure leads to pattern of behavior. Yet, beyond simple classroom examples and as guides to intuition, the concept has never been fully developed for large-scale systems with many loops. If the theory of how feedback loops lead to behavior can be developed to the point where it can be implemented as a computer algorithm, it would be of enormous help both in analyzing dynamics and explaining results.

Nathan Forrester (1983) suggested a method for relating the strengths (gains) of individual feedback loops to the system eigenvalues, which, in linear systems at least, capture most of what we mean by "patterns of behavior". The specific measure is the "eigenvalue elasticity", i.e., the relative change in an eigenvalue resulting from a relative change of the loop gain. A large elasticity would indicate that the loop is in some sense "important" in generating the behavior mode associated with that eigenvalue. So far, however, these ideas have not been developed further or applied to specific models. In the paper, we use tools from network theory and discrete mathematics to the problem and take some steps toward a computer algorithm that would automatically identify important feedback loops for a behavior pattern of interest and illustrate its use in a concrete model example.

Theory

We first address the largely unrecognized problem that even relatively small systems may contain a thousands or millions of feedback loops, as the number may grow more than factorially with the number of variables! Another way of looking at this is that the loop description of a system contains many redundancies. We analyze these interdependencies and show how only a relatively small number of loops can have their gains varied independently by an appropriate change in the strength of the links between variables. We prove a theorem stating that a "fully connected" feedback system with N links and n variables contains exactly $N-n+1$ independent loops. (A "fully connected" system is one in which there is a feedback loop between any pair of variables, as is normally the case in system dynamics models. Otherwise, the system may be decomposed into fully connected subsystems, where the theorem once again applies.) Since N typically grows linearly with n, and at most as n^2, this represents a substantial reduction in complexity.

We then present efficient algorithms for identifying all feedback loops in a system and for generating an independent subset. It is important to note that there is a large number of such sets to choose from, and we suggest ways to augment the algorithm to make it likely that the "important" loops (those with large eigenvalue elasticities) are selected.

Along the lines of Richardson (1995), we define the gain of a link $x \longrightarrow y$ as $\partial x/\partial y$ and the gain of a feedback loop as the product its constituent links. We show how the eigenvalue elasticities of links in the linearized system constitute a "current" in the network and that this current can be expressed as the sum of the independent loop elasticities ("loop currents"). Once the eigenvalues and eigenvectors of the system matrix have been determined, one can find the link and loop elasticities through simple linear matrix operations, which do not require extensive computer time.

Example: The Long Wave Model

We apply the procedure to John Sterman's simple long-wave model. That model was chosen both because it is wide known and has a relatively simple structure yet leads to complicated dynamics, and because nonlinearities play a key role in its dynamics, hence shedding some light on the applicability of the eigenvalue approach to nonlinear systems. The paper assumes that the reader is already familiar with this model; details can be found in Sterman (1985) and Haxholdt, et al. (1995).

The model contains a total of 36 feedback loops while the size of the independent set is 16. We illustrate how to select an independent loop set and then proceed to analyze the behavior of the model in terms of the shifting dominance of the loops in the independent set, most of which have an intuitive interpretation.

The strong nonlinear effects in the model are evident in large changes in both the gains and the eigenvalue elasticities of the loop gains. One may divide the long-wave cycle into four phases, which we will name "self-order growth", "capital growth", "self-order collapse", and "capital depreciation", respectively. In the first phase, the order backlog grows rapidly as a result of the self-ordering mechanism: Orders from the capital sector to itself swell the backlog, increasing the desired production capacity, which leads to yet more orders for capital. However, because capital orders are limited by a nonlinear function that sets a maximum for the expansion rate of the capital stock, the gain of the self-ordering loop drops close to zero after just 2-3 years! At this point, the system enters the second phase, where further growth in capital and orders is mainly determined by two positive loops: The "economic growth" loop reflects the standard physical capital accumulation in a growing economy. The "capacity expansion" loop arises because the nonlinear function in orders implies that orders are now proportional to the capital stock (as a "maximum fractional expansion"). After about 6 years, capacity catches up with desired output, and the loop gains once again shift back to their constellation in phase 1, as the system enters the third phase, "self-order collapse". Now, the positive loops work in reverse, quickly driving desired production down. Once again, however, the gain of the self-ordering loop is driven to zero in a few years, this time by the non-negativity constraint on orders. The system now enters the final phase, where capacity slowly falls and the system behavior is dominated by the single negative loop related to capital depreciation.

Although most of these results can be found by simply observing the shifting gains of the feedback loops over time and applying "old-fashioned" intuition, the eigenvalue elasticity analysis yields further insight into the role of individual loops. During the "self-order growth" phase, the rapid growth is attributable to a pair of complex conjugate eigenvalues with a positive real part. The time constant for the exponential growth is about 3 years while the period is too long (about 12 years) to play much of a role in the dynamics. The elasticity analysis clearly shows that the self-ordering loop plays a dominant destabilizing role in this mode. The main stabilizing influences come from the negative supply-line and capital correction loops, though both also increase the frequency of the eigenvalue. It is worth noting, however, that the loop gains and dominances shift very rapidly during the early expansion phase. For instance, the "overtime" loop is stabilizing at the very beginning of the phase, but quickly loses influence.

During the "capital growth" phase, the continuing growth is attributed to a single positive real eigenvalue with a time constant of about 5 years. Here, however, the self-ordering loop plays no role at all. Instead, the main destabilizing influence comes from the above-mentioned capital-expansion and economic-growth loops. Likewise, the overtime loop, though it is a negative loop, is destabilizing in this phase: Because growth is anchored on the increase in the capital stock, overtime allows for more output and hence faster capital accumulation, boosting the growth rate of the system. More surprisingly, the supply-line correction loop is also destabilizing. However, its destabilizing influence is largely canceled by a more obscure positive loop in the system that shares many of its links. This is an example of how the loop analysis may not always lead to simple intuitive results.

During the "self-order collapse" phase, there is once again an unstable pair of complex eigenvalues, with a time constant of about 2.5 years and a period of 35 years; the real part thus dominates the dynamics. As in the first phase, self-ordering plays a predominant destabilizing role, but the picture is complicated, as many other loops affect the eigenvalue, particularly its imaginary part. The "economic growth" loop is clearly destabilizing, as is the overtime loop; because they both increase the rate of capital accumulation, they exacerbate the peaking of capacity during the phase. As expected, the "capacity-expansion" loop plays no role during this phase, since orders are no longer anchored on capital stock. However, one must again be aware that loop gains and dominances shift very rapidly, so that a snapshot picture may not fully represent the whole time interval.

Finally, during the "capital depreciation" phase, most of the feedback loops in the system are shut off by the nonlinear functions that prevent orders, backlog, and supply line from falling below zero. The system now contains three real negative eigenvalues, the largest having a time constant of about 20 years, corresponding to the average lifetime of capital. The loop elasticities clearly show that three loops control the three modes: The 20-year decay is caused by the negative first-order capital-depreciation loop. The two other negative eigenvalues relate directly to the overtime loop (which acts as a first-order control on output preventing backlog from falling below zero), and a similar first-order control loop that controls shipments to the capital sector and prevents the supply line from becoming negative.

The system also contains a well-known positive loop, the "hoarding" mechanism, which arises because increased delivery delays increase the desired supply line, leading to further ordering and hence even longer delivery delays. However, the elasticity analysis clearly shows that this loop plays no role in the dynamics at any time, as may be confirmed directly by altering to model to base desired supply line on a constant normal delivery instead; with the hoarding loop thus disabled, the dynamics are virtually unchanged. The intuitive reason is that the hoarding mechanism is much weaker than self-ordering, so that it is simply not fast enough to "catch up" during the stages where the two loops are active. At other times, both loops are disabled by the nonlinearity in ordering.

Conclusion

The work presents a method which may be a step toward a systematic analysis of the role of feedback loops in system behavior. The most significant contribution is the notion of an independent loop set, which gives grounds for optimism about using the method to large-scale models, even though these will contain millions of feedback loops. Moreover, the model example demonstrates how the method can aid in getting a deeper understanding of the dynamics, even when the system is highly nonlinear.

That said, there is much room for improvement and further work. For instance, the eigenvalue analysis says nothing about how the behavior of individual variables are affected by feedback loops, which is particularly a problem when the eigenvalues are complex. To answer this requires considering the eigenvectors of the system, e.g. the participation factors (Eberlein 1984), but unlike the eigenvalues, the eigenvectors cannot be expressed as functions of loop gains only and one would therefore no longer have a clear definition of the relative importance of a particular loop. Clearly, this area warrants further attention. Likewise, a method for finding the "best" independent loop set, e.g. the one with the largest eigenvalue elasticity, ought to be developed, perhaps along the lines suggested in the paper.

Ideally, a number of alternative tools and representations ought to be available to the practitioner. The belief of the author is that many of these tools may be found in discrete mathematics, graph theory, and network theory. The final point of the paper is thus an appeal to fellow system dynamicists to look further into this discipline.

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