A DYNAMIC MODEL OF PORTFOLIO SELECTION: ANALYSIS OF BIFURCATIONS

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ABSTRACT

In this article we want to predict the dynamic behaviour of a portfolio of assets, i.e., we want to know how quickly it will move towards a new position of equilibrium when an unstable situation has ocurred due to important changes in the risk, (and in some cases in the return) of the securities.

In order to carry out this forecast, first we should locate the points of equilibrium, then analyse their stability and lastly determine where, and under what conditions, the discontinuities appear.

Changes in the return or in the risk of the securities which make up a portfolio can be smooth and this also brings about a smooth change in the portfolio, which is shown in a readjustment in its composition. However, it sometimes happens that, while the expectation of return remains relatively stable, circumstances arise which considerably increase the risk, in which case a serious discontinuity occurrs in the feature of the portfolio.

In this way, we can apply the methodology of Thom's Theory of Catastrophes in order to obtain valid conclusions using the morphology of the butterfly catastrophe for the portfolio's feature (efficient, non-efficient and opportunity or pseudo-efficient), employing four control factors: return, variance, transaction costs and risk aversion.

INTRODUCTION

A portfolio can be defined using the proportions x_i (i=1,...,n), of each of the securities which make it up over the total; from our point of view these proportions will be the variables of state. For each set of values of these variables which verifies with the condition $\sum_{i=1}^{n} x_i=1$ we will have the univocal definition of portfolio.

Let us consider to be n the total number of securities existing on the market, so that some of the x_i components may be null if the corresponding security does not form part of the composition of a determinate portfolio. In this way the number of securities remains fixed and equal to n, and a portfolio will be defined by a point in the space R^{n-1}. We thus remove the problem which would confront us if we incorporated or dispense whith some securities and this would be reflected in the change in the dimension of the support space.

Therefore, the variables of state x_i, which are positive or null and verify $\sum_{i=1}^{n} x_i=1$, itemize completely the composition of the portfolio and in consequence, define it. From this definition, it is obvious that a single security can make up a portfolio.

When portfolios are analysed (both in the MPT as in the CAPM) it is considered that the elements which characterise them (control variables) are:

- expected returns
- risk inherent in these returns

Basing ourselves on these two elements, portfolios can be classified into efficient and inefficient. A portfolio is efficient if, given a certain return, no other portfolio exists with an equal return with has a lower risk level, or inversely, if a portfolio exists with a certain level, there is no other wich , having exactly the same risk, offers higher returns.

We are not going to get into a discussion on the methods used to optimise and consequently select the portfolio (see for example Sharpe,(1950), as this is essentially timeless, conforming at the most to a model of comparative dynamics; and the few dynamic models which have been formulated cannot be considered as more than basic "stochastic" formulae which are outside the area this study is focussing on.

Amongst other things, our analysis is an attempt to study the well known situation that occurrs when an efficient portfolio no longer is so, due to objective causes on the stock market.

As a result, it is necessary to consider that, with time, efficient porfolios constitute equilibrium which under certain circumstances, can lose its stability, giving way to a new state of the system.

We therefore need to have a model which explains the discontinuities and changes in the surface of equilibrium. This model must be implicitly dynamic and we think it appropriate to use the posibilities offered by the catastrophe theory.

Of all the restrictive hypotheses on which the analysis of portfolios is based (MPT, CAPM,...), there are two which we do not need to include in the present work.

In the first place, we are not going to restrict ourselves to the limitation of the temporal horizon. "That single, common horizon allows us to construct a single-period model. The model implies that investors buy all the assets in their portfolios at one point of time and sell them at some undefined but common point in the future." (D.R. Harrington, 1983). This hypotesis, is in any case totally unreal and is unnecessary if we carry out the dynamic analysis using the methods provided by the catastrophe theory.

Also, we will include the possibility of considering transaction costs in our analysis, which are not taken into account in the present form of the portfolio selection theory.

The relaxing of the CAPM model hypotheses permits the treatment of situations such as:

- An inefficient portfolio could becaming efficient in time (or become more inefficient).
- The investor, covering the transaction costs, can convert an inefficient portfolio into an efficient one (or this is at least what he hopes) by modifying its composition.

With the conclusions we reach from analysing situations such as the ones mentioned, we are able to study under which circumstances all the efficient portfolios support the buffetings of the market and pass from a state of equilibrium to another, determining the type of equilibrium achieved under the new situation.

There is no reason to believe that only one path of behaviour exists as, whether the recovery of an efficient equilibrium is achieved by means of transaction (buying and selling securities) or by a situation on the market, different points can be reached by different paths (hysteresis).

TOPICS ON THE ANALYSIS OF PORTFOLIOS.

The first model for the selection of portfolios was developed by H. Markovitz in 1952. Basically, it postulates that the investor should choose, from between all the possible combinations of investments, the one which minimizes risk and maximizes the expected return.

As it concerns an optimisation with more than one objective, this is reduced by arrangement of portfolios according to yield, and within those with the same level of return, according to their level of risk.

If we use E to represent the expected return and s as an adequate measurement for the risk (normally the standard desviation of the expected returns in the portfolio), the set of portfolios denominated as efficient can be represented by figure 1, (Sharpe, 1950).

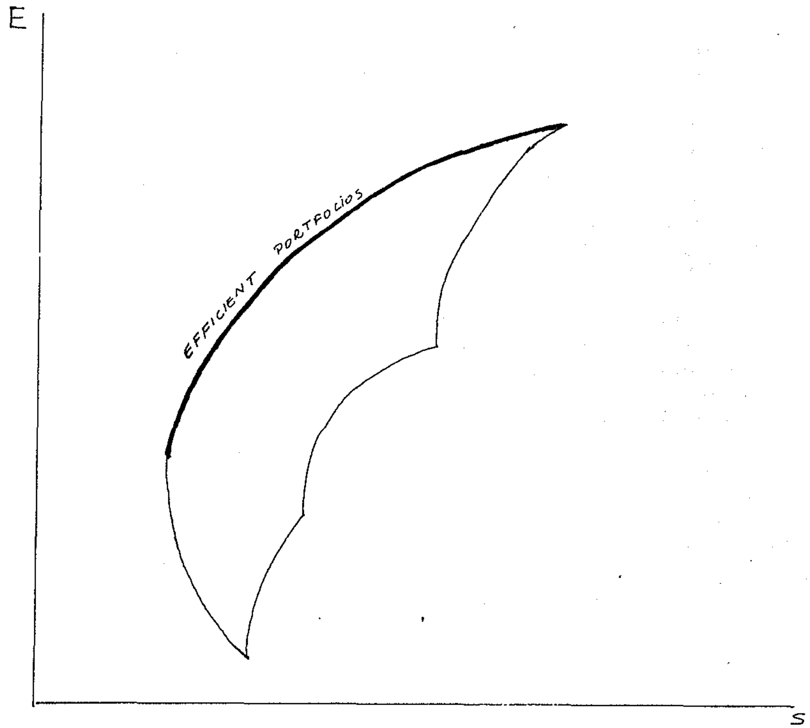

Figure 1. The efficient set and boundary of portfolios.

A portfolio's return will be weighted average of the yields of each one of the securities, using as a weighting factor the proportion of each security in the portfolio.

The return, or profitability expected from a security, normally referred to a period (economic exercise), is determined on the basis of future expectations and using past experience as a reference. Usually, this formula is used:

$$\text{Return} = \frac{\text{Dividends} + PM_t - PM_{t-1}}{PM_{t-1}}$$

where PM_t is the market price at the moment t.

Somewhat more complex are the concept and the measurement for the risk.

The introduction of the variable risk in the definition of an efficient portfolio is justified by the differences in individual behaviour when faced with risk.

- If the individual is completely averse to risk, he will select a portfolio with no risk involved (made up of fixed-income securities), even at the cost of a possible low return, or else he will invest all his capital in risk-free asset.
- If he is a risk taker, he fully enters into Markovitz's analysis as he could be an investor for a certain level of risk/yield but not for another higher one.
- If he is a risk lover, he will choose those investments with a high expected yield, without bothering about the level of risk.

As the greater part of the "normal" investors belong to the second category, since when putting together a portfolio they seek to control the risk (although not eliminate it), we need a form of measurement for the risk.

Normally, risk is identified with the variability of return. In a natural way, this suggests that the variance or standard desviation of the expected value of the profitability be used as the measurement of risk. In practice, these measurements have shown their use as opposed to others which appear in literature concerning portfolio analysis (D.R. Harrington, 1983).

The suggested measurement of risk is sufficient for analysis of an isolated security. However, the risk of a portfolio cannot be measured only as the weighted total of the risk of the yields on each of the securities, a relation which can be measured by its correlation or by its covariance.

But when the portfolio has a certain number of securities, the excessive number of calculations necessary makes the practical application of this method doubtful.

To remove this problem, different versions of Markovitz's model have been developed, one of them being the CAPM (Capital Asset Pricing Model). Instead of correlating the different securities amongst themselves, a point of reference known as the " market portfolio " is chosen, which is defined as the portfolio formed of all the securities present in the market, their participation in the portfolio being the same as it is on the stock market.

A portfolio's risk is then defined as the covariance of the stock return with respect to the market return, that is to say, the only risk which is taken into account is the systematic one; while the non-systematic risk of each security is eliminated by the formation of the portfolio.

"The CAPM rests on eight assumptions. The first five assumptions are those that underlie the efficient-market hypothesis and thus underlie both MPT and the CAPM. The last three assumptions are necessary to create the CAPM from MPT.

The eigth assumptions are the following:

1. The investor's objective is to maximize the utility of terminal wealth.
2. Investors make choices on the basis of risk and return. Return is measured by the variance of these portfolio returns.
3. Investors have homogeneous expectations of risk and return.
4. Investors have identical time horizons.
5. Information is freely available to investors.
6. There is a risk-free asset, and investors can borrow and lend at the risk-free rate.
7. There are no taxes, transaction costs, or other market imperfections.
8. Total asset quantity is fixed, and all assets are marketable and divisible."

(D.R. Harrington, 1983, p.22).

Based on all these hypotheses, the CAPM model allows the investor to select the optimum portfolio according to his preferences.

The model has a very extensive use and some of the market hypotheses on which it is supported have been conveniently contrasted. But there are others whose general validity is doubtful; in particular for us hypoteses 3, 4 and 7.

THE CATASTROPHE'S METHODOLOGY AND THE ANALYSIS OF PORTFOLIOS.

The problem we want to solve is the one which crops up when the investor who possesses a portfolio which up to a certain moment has proved to be efficient, is faced with the fact that other portfolios exist with a higher return and the same level of risk, or a lower risk and the same return. In other words, his portfolio is no longer located on efficient boundary.

It is obvious that this can only happen over a period of time and is due as much to typical stock market operations as to the performance of the actual companies issuing the securities who modify not only the expectations on which the relation return-risk is based but also the actual economic reality.

If the investor has chosen the portfolio according to some criterion based on economic rationality and considers it to be optimum within its utility function, and sees, as time goes by, that becomes less efficient, we must examine what kind of equilibrium it must have reached at its peak and carry out a qualitative and quantitative analysis of the behaviour of portfolios.

We are going to carry out this analysis using the techniques offered by the catastrophe theory as we think there is sufficient theoretical and empirical support with which to tackle this study on portfolio selection from this point of view, and because we think that in this way a dynamic approach to a portfolio's performance can be given.

We should point out that the selection of portfolios complies with various requirements which, according to the writers who have applied the methodology of the catastrophe theory of the social sciences, are necessary for the application of the said methodology to be considered adequate. These requeriments are:

BIMODALITY: the portfolios are efficient or inefficient and either of them produces a stable equilibrium.

HYSTERESIS: we have already commented on the appearance of this phenomenon during the process of determining the efficiency of portfolios.

SUDDEN JUMPS: it is obvious that these occur and may be due to operations involving capital, cuts in dividends and even for reasons outside the market which are the most important, given the frequency with which they occur and the unpredictability of their appearance.

DIVERGENCE: (as regards the risk associated with securities returns) since two points united by neighbouring paths can produce perceptibly different behaviour.

Some authors shown that the existence of only some of these requirements justify the analysis of dynamic behaviour by means of the catastrophe theory. In our case, not only are all the requirements fulfilled but others, which are more important from the point of view of theory, are also verified. For instance, the direct application of the Morse lema and the splitting theorem.

In fact the investor assumes that must choose the proportions of assets x_i which must become part of the portfolio, in such a way that, at a given level of return they minimize the risk, or, with an acceptable level of risk, they maximize the expected return, taking into account that he always wishes to maximize the value of the final wealth.

The optimum obtained is isolated, which allows us to use the Morse lemma and the splitting theorem not only in the selection of one portfolio but for the whole portfolio family.

Let us begin then, although briefly, with a review of the basic elements on which the theory of catastrophes rests and then we shall see whether the selection of portfolios can be included in any of the seven elementary catastrophes in accordance with Thom's classification theorem.

It is not our intention to modify the portfolio by considering the proportions of x_i but to base our analysis on the performance of the portfolio through the control variables we choose, such as its return or risk, measured in the same way as in the MPT or CAPM or any other models used for the selection of portfolios.

We will consider a family of functions defined by

$$V: P \times C \longrightarrow R$$

in which P is the manifold of the possible portfolio and as we have seen $P \subset R^{n-1}$ (including 0) and which we have named space of variables of state.

C is another manifold $C \subset R^r$ which we will call space of control factors.

As we have seen, the analysis of portfolio selection is based on the binary return-risk and determines the composition $x=(x_1,...,x_n)$, $\sum x_i=1$ assuming that the investor wishes to maximize the value of his wealth at the end of the period.

This gives an optimum portfolio, valid for a period which we can designate by $x=x(E,s)$, and we are not only interested in knowing under what conditions this portfolio is no longer optimum, but also when it will lose its efficiency.

The methodology of the catastrophe theory assumes that, locally, a portfolio's performance is regulated by the potential $V(x;E,s)$. In a more general way, V is taken to be the function of x and all the control variables which are involved in the process. Therefore the potencial which regulates the performance of the process can be shown as $V_c(x)=V(x,c)$.

If V is the universal unfolding then the set of catastrophes M, which is also a manifold, is the subset of $R^{n-1}xR^r$ defined by:

$$DV_c(x)=0$$

In other words, the set of all the critical points of all the potentials V_c in the family V. The set of catastrophes M will be a manifold if V, which also is, represents the universal unfolding.

By restricting M, mapping the natural projection, we obtain a map χ defined by:

$$\pi: R^{n-1}xR^r \longrightarrow R^r \qquad \text{being } \pi(x,c)=c$$

starting with χ, which is known as the catastrophe map, we obtain the set of singularities S, which will be the set of singular points in M, according to χ, for those for which χ is singular, i.e. that the range of the Jacobian matrix D_χ is less than r (dimension of the space of control variables).

The image of the sets in C, according to the map χ, is known as the BIFURCATION SET B. It is evident that S is the set of degenerate critical points.

B is called bifurcation set because it is the set over which the number and nature of critical points change and, considering the structural stability of the Morse functions, such changes can only take place by passing through a degenerate critical point.

THE CATASTROPHE IN CUSP

If we take return and risk as control variables, the transformation, via a diffemorphism, from the families of portfolios into a potential function for universal unfolding will contain only two parameters. Among the universal unfoldings of Thom's seven elementary catastrophes, there is only one with two parameters of control, which is called catastrophe in cusp.

The universal unfolding is:

$$V(x) = x^4+ux^2+vx$$

The surface of equilibrium M, which allows a representation in the ordinary space, is given by the equation:

$$V'(x) = 4x^3+2ux+v = 0$$

and the degenerate critical points (singularities) are given, in the same way, by the subset of M which satisfies:

$$V''(x) = 12x^2+2u = 0$$

If we eliminate x from the last two equations, we obtain:

$$8u^3+27v^2 = 0$$

This equation determines the bifurcation set B, the set of all points of the control space C(u,v) in those where the form of V changes.

Its representation is shown in figure 2, and we have also pictured the stability of equilibrium. Out of the bifurcation set there is only one point of stable equilibrium (minimum potential): this means that the portfolio will either be efficient or inefficient, with any other intermediate position. Within the bifurcation set (set of degenerate critical points) there are three points of equilibrium (two minimums separated by a maximum), two stable and one unstable.

With reference to the behaviour surface of portfolios (figure 3), and taking into account that the analysis carried out is local, the majority of authors on the catastrophe theory interpreted the area (shadowed) corresponding to the bifurcation set as an inaccessible region.

In our model this area is significant, in the sense that if the portfolio is located in this region, any disturbance, however small, will take it to the higher surface (efficient portfolios) or to the lower one (inefficient portfolios).

Also, in figure 3 we represent the bifurcation set, that is the projection of the surface on to the control space (u,v). The projection of the surface folds define the boundary of the

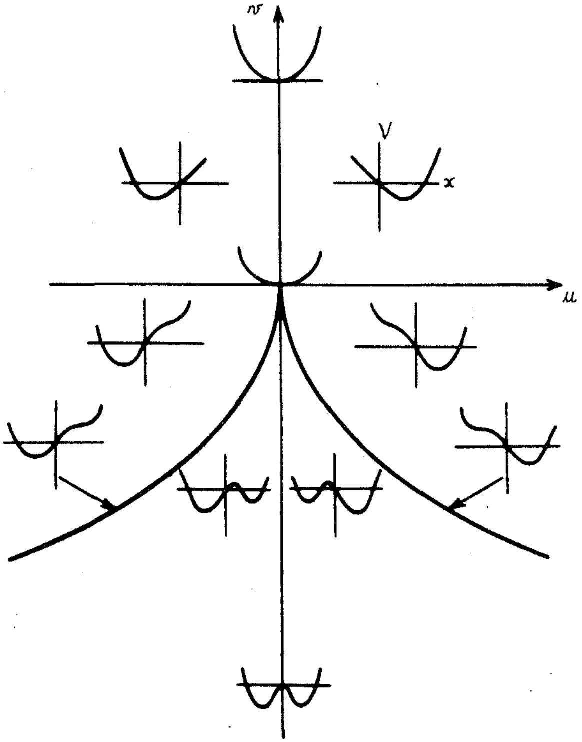

Figure 2. Situation and number of the equilibrium points and their stability in the cusp catastrophe. (Saunders 1983, p.12)

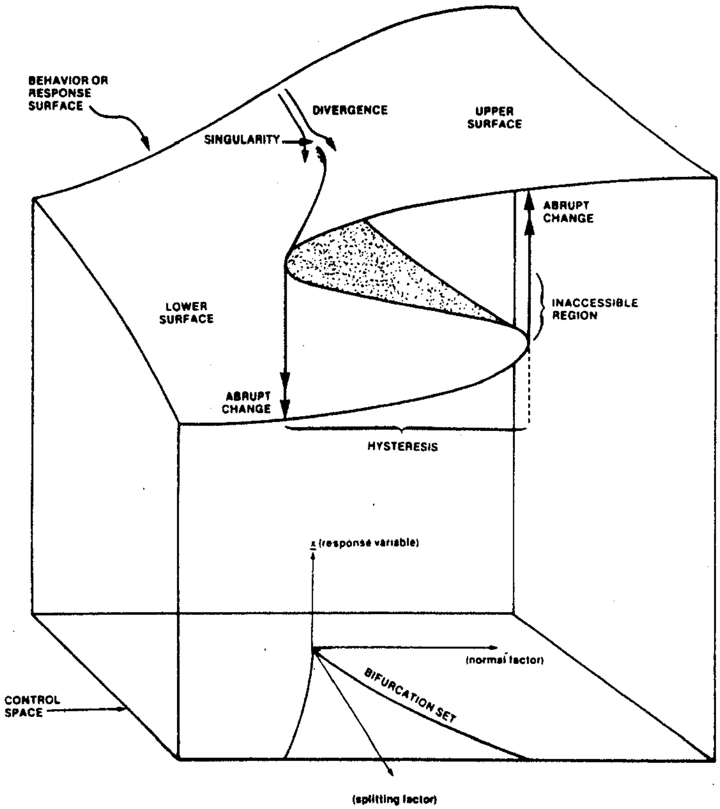

Figure 3. The behaviour surface and this bifurcation set of the cusp catastrophe. (Flay 1.978, p. 337)

bifurcation set.

Parameters of the control space u and v are called separation factor and normal factor respectively. In our case we have identified them with risk and return of a portfolio. On the upper right-hand part of the surface lie portfolios preferred by investors, the efficient ones, characterized by high return and low risk. If return falls and level of risk remains acceptable, the portfolio becomes inefficient following a smooth path, such as a-b-c-d (figure 4). Nevertheless, when risk increases and return decreases following the a-e-d path, i.e. in an unsmooth way, there is a sudden jump to the surface representing inefficient portfolios. (Delay perfect convention).

Since the variables u (risk measured by means of variance) and v (return) can be quantified, bifurcation set can easily be settled for each pair (u,v). The region of unstable equilibrium is then univocally determined and consequently the boundary between the set of efficient and inefficient portfolios.

We would like to point out that, similarly to our analysis of portfolio selection, our approach is only valid for those investors having a "normal" risk-aversion. That is, both for those who are totaly averse to and for those who love taking risks, this analysis is either partial useful or useless. This fact restricts this model's field of application and although the group of investors remaining outside it is small, it is still important, and for this reason we will consider a wider approach embracing this set of investors.

There is another problem, in our opinion more important than the above, and that limits the application of this model: the catastrophe's theory states that once a new stable equilibrium is reached and it is a not convenient situation, one has to remake all the analysis.

In our case this means that the investor who has an inefficient portfolio should keep it, since that in order to transform it into an efficient one he would have to pay transaction costs which, as we have seen, are not allowed in the analysis of portfolio selection.

The possibility of the re-investing profits (either from capital or from dividends) also remains outside analysis.

These problems are due to the fact that we are carrying out a dynamic analysis of a situation that at most will take us to comparative statics within the traditional analysis of portfolio selection.

In order to get out of this impasse, we must establish a catastrophe model which, while maintaining our two control variables and acting in the same way in the potential that

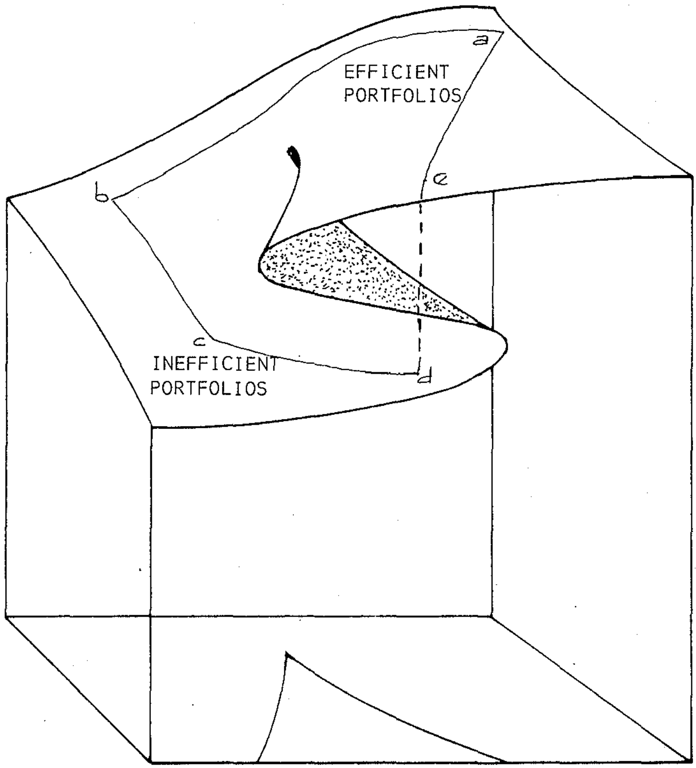

Figure 4. Two different paths leading to the same point: one smooth and other one with sudden jump.

regulates the process, amplifies our field of action as far as the number of control factors is concerned; i.e. we choose a control space of greater dimension.

We face two possible catastrophes: the swallow tail and the butterfly. For reasons that will become clear in the following exposition, we are carrying out the analysis and application of the butterfly catastrophe.

THE BUTTERFLY CATASTROPHE

The universal unfolding is in the form:

$$V(x) = x^6+tx^4+ux^3+vx^2+wx$$

where t, u, v, w, are the control factors. As we can see, the control space has dimension 4, and therefore we cannot represent bifurcation set B.

There are several ways to approach the study of the form of the bifurcation set, but the one leading a more evident graphic was carried out by Brocker and Lander (1975), Saunders (1980), and, Poston and Stewart (1978). We are following the analitic development formulated by Saunders.

Initially, and considering t and u as constants, we study the form of the different subsets B_{tu}, which will provide us with a great deal of information about the bifurcation set.

The surface of equilibrium M is:

$$6x^5+4tx^3+3ux^2+2vx+w = 0$$

The set of degenerate critical points (singularities) is the subset of M which also verifies the following equation:

$$30x^4+12tx^2+6ux+2v = 0$$

This system of equations allows us to define v and w as a function of the other variables, and which therefore determines, fixing t and u, the B_{tu} curves. Thus, e.g., for t = u = 0, we obtain as an equation for B_{tu}:

$$-(u/15)^5 = (v/24)^4$$

then B_{oo} is a cusp.

To analyse the behaviour of B for other values of t and u, different from zero, we must consider v=v(x) and w=w(x), in such a way that its critical points satisfy an equation of the type:

$$20x^3+4tx+u = 0$$

which will always have a real root, and therefore B_{tu} will have at least one cusp point, and 3 if all the roots are real; that is to say if they verify:

$$u^2+4(4t/3)^3 < 0$$

This condition cannot be satisfied if t is positive and it is for this reason that t is called the BUTTERFLY FACTOR, because we really have a catastrophe of this type when this factor varies, and then we are able to move from one to three cusps.

So, when the control factor t of the butterfly catastrophe is positive the hypersurface of unfolding of this catastrophe is very similar to the surface of unfolding in cusp; the only difference is that instead of being folded in curves, it is folded along whole surfaces.

The control factor u is known as the BIAS FACTOR, due to the fact that B_{tu} is symmetrical to the axis v only if u = 0. This factor, which plays the same role in the swallow tail catastrophe, is not very significant for our purposes if we consider it without the butterfly factor.

As before, w and v, are still denominated normal and separation factors, and they have the same significance and role as the in the case of a cusp catastrophe.

If we assume that the butterfly factor has been previously fixed and is negative, and that the bias factor vanishes, the graphic representation of the behavioural surface of the butterfly catastrophe would be similar in form to that in figure 5, in which the control space for the control factors (v,w) has been projected.

We should point out that a third surface exists between the two had in the cusp catastrophe. Our interpretation of this surface is as follows:

If an investor has an inefficient portfolio he can change and go from this to an efficient portfolio by means of the sale and/or purchase securities. But this incurs expenses. As a result, the butterfly factor which, as we have seen, must be negative for this third surface to appear, is interpreted in transaction costs.

If the investor finds that he has a portfolio which is on the third surface, he will be faced with the following problem:

> "I am interested in incurring transaction costs (buying and selling) provided that the additional return that I obtain as a result of moving up to the set of efficient portfolios is greater than these costs; I do not mind if they are the same and if the costs are greater than the

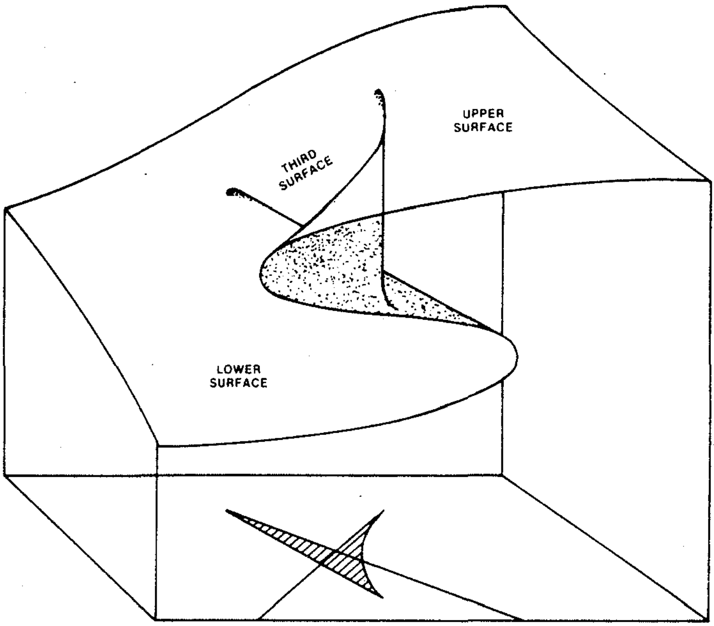

Figure 5. The butterfly catastrophe when the butterfly factor t is negative and the bias factor is null. (Flay 1978, p. 339)

expected return I will remain on the third surface".

For this reason we can say that the third surface is made up of portfolios which, not being wholly efficient, do not oblige the investor to change the composition of the portfolio, due to the fact that the additional return would not compensate the trasaction costs. They therefore represent a compromise solution, a waiting period until a good opportunity arises. These portfolios are thus called pseudo-efficient portfolios (opportunity portfolios).

We could apply the same interpretation to the case where the investor is waiting for an opportunity to invest surplus, as he does not consider the present market situation to be adequate. This could be the result of finding himself in an intermediate situation (between efficient or inefficent portfolios) of a relatively stable equilibrium; at least quite a lot more stable than in the cusp catastrophe.

In any case, the area of surface stability, furthermore to depend the negative values of t, also relies basically on the fact that the bias factor u and the separation factor v have values near zero.

This can be seen more clearly if we represent separately the projection of one section of the bifurcation set. In figue 6 we have put together the representation of this group based on the supposition that u=0, t=0 (both fixed). In this figure the number of equilibrium points reached is also indicated as well as their stability.

In figure 7 we can see the evolution of the section of the bifurcation set for different values, positive and negative, of t and u. We must mention that the "pocket" appears when the bias factor is null and the butterfly factor negative.

We are now in conditions to give the correct interpretation of the bias factor u, according to our model. Above we mentioned the need to apply the model only to those investors who behave in a normal way in front of risk. In our model the possibility of possessing an opportunity portfolio, in other words to find the portfolio on the third surface, is only possible for those individuals who are not biased, meaning that they have a null value of control factor u.

On the other hand, both the positive values and the negative values of the bias factor u make the third surface disappear from the set of portfolios (even in the event of t being negative) and in the projection in the space of the control variables the characteristic "pocket" disappears from the butterfly catastrophe as can be seen in figure 7.

Also in figure 8, we have the graphic representation of the behaviour's surface, assuming that the bias factor takes either negative and positive values. The positive bias factor

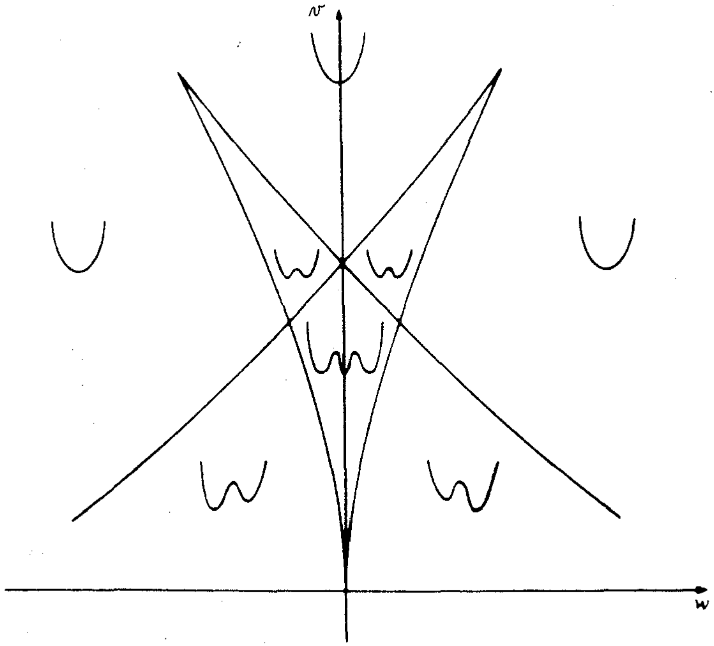

Figure 6. Transversal section of the bifurcation set of butterfly catastrophe when u=0, t< 0. There are drawn the situation and number of the equilibrium points and their stability. (Saunders 1983, p. 68)

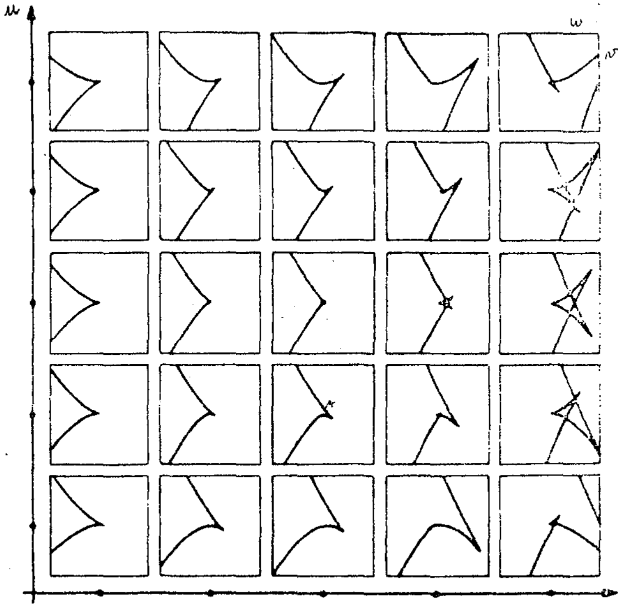

Figure 7. Butterfly catastrophe. Section corresponding to various values of t and u, viewed in the (v, w)-plane. (Poston and Stewart, 1981, p. 178)

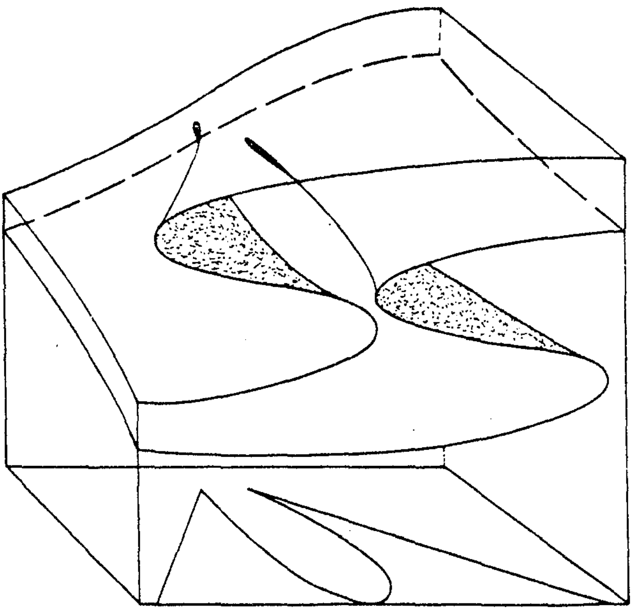

Figure 8. The butterfly catastrophe surface when the factor t is negative, showing the effects of bias u. (Flay 1978, p. 338)

can be the behaviour of an investor "risk-lover" and the negative bias factor the behaviour of the "total risk-averse". Finally, the butterfly catastrophe model correctly explains the behaviour of an investor who faces the risk in a normal way, that is, he is not biased by the risk.

CONCLUSIONS

In this article I have tried to eliminate some limitations of traditional analysis in portfolio selection. Specifically:

- The model is a dynamic one
- I have taken into consideration as well as the usual variables (return and variance), transaction costs and attitudes of investors when facing risk.

In order to extend this model I have used Catastrophe Theory methodology, taking the portfolios as the behaviour surface and the expected return and the variance as control variables.

I have shown that:

- the analysis can be done using the catastrophe in cusp as portfolio morphology
- this model is insufficient to reach my purposes
- using four control variables (return, variance, transaction costs and attitude facing risk), the behaviour model corresponding to the butterfly catastrophe
- in either case the bifurcation set of portfolios is clearly defined, and,
- the investors can choice one portfolio (or a portfolios family) in according with their preferences.

Barcelona, june 1986

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